Problem sheet 1

1. Lagrange interpolation formula

Let x_0, x_1, ..., x_n be $(n+1)$ pairwise distinct points and let there be given $(n+1)$ arbitrary numbers y_0, y_1, ..., y_n. Further define the fundamental Lagrange polynomials by

$$l_i^n(x) \quad = \quad \frac{(x-x_0)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}.$$

(a) Show that

$$p_n(x) \quad = \quad \sum_{i=0}^{n} y_i l_i^n(x)$$

is the unique polynomial $p_n(x)$ of degree n such that

$$p_n(x_i) = y_i, \qquad \text{for } \ i = 0, 1, ..., n.$$

Hint: You may first show that $l_i^n(x_j)$ equals 1 if $i = j$ and zero if $i \neq j$.

(b) Let $\Pi(x) = (x-x_0)(x-x_1)...(x-x_n)$ and show that

$$p_{(2n+1)}(x) \quad = \quad \sum_{i=0}^{n} \left[f(x_i) \left(1 - \frac{\Pi''(x_i)}{\Pi'(x_i)}(x-x_i) \right) + f'(x_i)(x-x_i) \right] \left(l_i^n(x) \right)^2$$

is the unique polynomial of degree $(2n+1)$ for which

$$p_{(2n+1)}(x_i) = f(x_i), \quad p'_{(2n+1)}(x_i) = f'(x_i), \quad \text{for } \ i = 0, 1, ..., n.$$

2. Orthogonal polynomials

Let $P_0(x)$, $P_1(x)$, ... be a set of orthonormal polynomials, e.g.

$$\int_a^b dx\ w(x) P_i(x) P_j(x) \quad = \quad \delta_{ij},$$

let x_1, x_2, ..., x_{n+1} be the zeros of $P_{n+1}(x)$ and w_1, w_2, ..., w_{n+1} the corresponding Gaussian weights given by

$$w_j \quad = \quad \int_a^b dx\ w(x) l_j^{n+1}(x).$$

Show that for $i, j < n+1$

$$\sum_{k=1}^{n+1} w_k P_i(x_k) P_j(x_k) \quad = \quad \delta_{ij},$$

e.g. the P_0, P_1, ..., P_n are orthonormal on the zeros of P_{n+1}. This equation can be useful to check the accuracy with which the zeros and weights of P_{n+1} have been determined numerically.

3. Monte Carlo integration

The Monte Carlo estimate for the d-dimensional integral

$$I = \int dx f(x) = \int d^d u f(u_1, ..., u_d)$$

is given by

$$E = \frac{1}{N} \sum_{n=1}^{N} f(x_n).$$

In order to discuss the error estimate for finite N, we first introduce the variance $\sigma^2(f)$ of the function $f(x)$:

$$\sigma^2(f) = \int dx \left(f(x) - I\right)^2$$

Show that

$$\int dx_1 ... \int dx_N \left(\frac{1}{N} \sum_{n=1}^{N} f(x_n) - I \right)^2 = \frac{\sigma^2(f)}{N},$$

e.g. the variance with which the Monte Carlo estimate differs from the true value is given by $\sigma^2(f)/N$.
Hint: You may want to introduce an auxiliary function $g(x) = f(x) - I$ and show $\int dx g(x) = 0$ first.

4. Pseudo-random numbers

Pseudo-random numbers were in the early days often generated by a multiplicative linear congruential generator according to the prescription

$$s_i = (a s_{i-1} + c) \bmod m$$

Consider the simple multiplicative linear congruential generator with $a = 5$, $c = 1$, $m = 16$ and $s_0 = 1$. Write down the first twenty numbers generated with this method. How long is the period? Write down the sequence also in the binary representation and look at the lowest order bits.

5. Gray code

The Gray code representation for a number n is a binary representation such that the representations for n and $n + 1$ differ in only one bit. The Gray code representation can be obtained from the binary representation according to

$$...g_3 g_2 g_1 = ...b_3 b_2 b_1 \oplus ...b_4 b_3 b_2$$

where $\oplus$ deontes the bitwise exclusive-or operation ($0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$). Find the Gray code representation for $n = 0, 1, ..., 7$.