Introduction to Theoretical Elementary Particle Physics:

Relativistic Quantum Field Theory

Part I

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Contents

1 Overview

1.1 Literature

There is no shortage of text books on quantum field theory. I will list a few of them here:

- M. Peskin und D. Schroeder, An Introduction to Quantum Field Theory, Perseus Books, 1995.
- M. Schwartz, Quantum Field Theory and the Standard Model, Cambridge University Press, 2014.
- M. Srednicki, Quantum Field Theory, Cambridge University Press, 2007.
- D. Bailin und A. Love, Introduction to Gauge Field Theory, A. Hilger, 1986.
- T. Muta, Foundations of Quantum Chromodynamics, World Scientific, 1987.
- M. Böhm, A. Denner and H. Joos, Gauge Theories of the Strong and Electroweak Interactions, Teubner, 2001.
- C. Itzykson and J.-B. Zuber, Quantum Field Theory, McGraw-Hill, 1980.
- J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics, McGraw-Hill, 1964.
- J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields, McGraw-Hill, 1965.

1.2 Units

It is common practice in quantum field theory and elementary particle physics to use natural units. Thus, by convention we set

$$\hbar = c = 1.$$

For example, the equation

$$E^2 - c^2 \vec{p}^2 = m^2 c^4$$

simplifies to

$$E^2 - \vec{p}^2 = m^2.$$

We will use this convention in these lectures. To facilitate the transition to this convention, we will still write in the very beginning of this course the quantities $\hbar$ and c explicitly, but soon set them equal to one as we proceed.

Energy is measured in eV:

$$1 \text{ eV} = 1.6021764 \cdot 10^{-19} J.$$

The following prefixes are used:

$$\begin{aligned} 1\text{ keV} &= 10^3\,\text{eV}, \\ 1\text{ MeV} &= 10^6\,\text{eV}, \\ 1\text{ GeV} &= 10^9\,\text{eV}, \\ 1\text{ TeV} &= 10^{12}\,\text{eV}. \end{aligned}$$

Momenta are measured in $\text{eV}/c = \text{eV}$, masses are given in $\text{eV}/c^2 = \text{eV}$.
From the uncertainty relation

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

it follows that lengths are given in $\hbar c \cdot \text{eV}^{-1} = \text{eV}^{-1}$.
Cross sections are given in barn:

$$1\text{ barn} = 10^{-28}\text{ m}^2.$$

Commonly used prefixes in particle physics are:

$$\begin{aligned} 1\text{ nbarn} &= 10^{-9}\,\text{barn}, \\ 1\text{ pbarn} &= 10^{-12}\,\text{barn}, \\ 1\text{ fbarn} &= 10^{-15}\,\text{barn}. \end{aligned}$$

Conversion constant:

$$(\hbar c)^2 = 0.389379292 \cdot 10^9\text{ GeV}^2\text{ pbarn}.$$

This is the most important conversion constant. Typical experiments are scattering experiments. The momenta of the incoming particles are usually given in GeV. If all calculations are performed in units of GeV, then the cross section has units GeV^{-2}. The conversion constant above converts it to pbarn.

1.3 The fundamental forces

We know four fundamental forces: the strong force, the weak force, the electro-magnetic force and the gravitational force. Particle physics deals with the strong, the weak and the electro-magnetic force. The gravitational force is negligible against the other three forces at present energy scales.

The standard model is based on a local gauge theory with gauge group

$$SU(3) \times SU(2) \times U(1)$$

$SU(3)$ corresponds to the strong interactions, $SU(2)$ to the weak isospin, $U(1)$ to the hypercharge. The symmetry of the subgroup $SU(2) \times U(1)$ is spontaneously broken down to the familiar $U(1)_{\text{el-magn}}$ symmetry of electro-magnetic interactions.

1.4 The elementary particles

The spin of the particles:

- **Fermions** have half-integer spin. In the standard model all fermions have spin $1/2$. In extensions of the standard model higher spins may occur, e.g. the gravitino with spin $3/2$.

- **Bosons** have integer spin. In the standard model all bosons have either spin 1 (gauge bosons) or spin 0 (Higgs boson). In extensions of the standard model there might be particles of higher spin, e.g. a graviton of spin 2.

1.4.1 Spin 1/2 particles

Quarks: Quarks feel the strong, the weak and the electro-magnetic forces. There are six quarks:

up, $Q_u = \frac{2}{3}$ $m_u < 10\,\mathrm{MeV}$	charm, $Q_c = \frac{2}{3}$ $m_c = 1.15 - 1.35\,\mathrm{GeV}$	top, $Q_t = \frac{2}{3}$ $m_t = 174 \pm 5\,\mathrm{GeV}$
down, $Q_d = -\frac{1}{3}$ $m_d < 10\,\mathrm{MeV}$	strange, $Q_s = -\frac{1}{3}$ $m_s = 80 - 130\,\mathrm{MeV}$	bottom, $Q_b = -\frac{1}{3}$ $m_b^{\overline{\mathrm{MS}}} = 4.1 - 4.4\,\mathrm{GeV}$ $m_b^{1\mathrm{S}} = 4.6 - 4.9\,\mathrm{GeV}$

The different quark types (up, down, strange, charm, bottom, top) are called "**flavours**".

Leptons: Leptons do not feel the strong interaction. There are six leptons:

ν_e, $Q_{\nu_e} = 0$ $m_{\nu_e} < 3\,\mathrm{eV}$	ν_μ, $Q_{\nu_\mu} = 0$ $m_{\nu_\mu} < 0.19\,\mathrm{MeV}$	ν_τ, $Q_{\nu_\tau} = 0$ $m_{\nu_\tau} < 18.2\,\mathrm{MeV}$
e, $Q_e = -1$ $m_e = 511\,\mathrm{keV}$	μ, $Q_\mu = -1$ $m_\mu = 105.7\,\mathrm{MeV}$	τ, $Q_\tau = -1$ $m_\tau = 1.78\,\mathrm{GeV}$

Neutrinos are electrically neutral and interact only through the weak force.

The family structure of the standard model: The fermions can be grouped into three families:

$$\begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} c \\ s \\ \nu_\mu \\ \mu \end{pmatrix}, \quad \begin{pmatrix} t \\ b \\ \nu_\tau \\ \tau \end{pmatrix}.$$

The families differ only by the masses of their members.

1.4.2 Spin 1 particles

Within the standard model the mediators of the interactions are spin 1 particles.

The strong interaction: SU(3): The gauge group $SU(3)$ describes the strong interaction. The number of generators for a group $SU(N)$ is $N^2 - 1$, therefore there are 8 generators for $SU(3)$, and hence 8 gauge bosons for the strong interactions. The gauge bosons of the strong interaction are called **gluons**. The fields are denoted by

$$A_\mu^a,$$

where a runs from 1 to 8.

The weak isospin: SU(2): The weak interaction is described by the gauge group $SU(2)$. There are three generators

$$W_\mu^1, W_\mu^2, W_\mu^3,$$

each with two polarisation states. After electro-weak symmetry breaking we use the fields

$$W_\mu^+, W_\mu^-, Z_\mu,$$

with three polarisation states. We have the following relations:

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right), \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_w W_\mu^3. \end{aligned}$$

The third spin degree of freedom comes from the Higgs mechanism.

The hypercharge: U(1): The last piece of gauge-symmetries within the standard model is given by an abelian $U(1)$ gauge symmetry, the hypercharge. The field is denoted by

$$B_\mu$$

After electro-weak symmetry breaking the photon field A_μ is given as a linear combination of B_μ and W_μ^3:

$$A_\mu = \cos\theta_W B_\mu + \sin\theta_w W_\mu^3.$$

Note that

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}$$

and that A_μ remains a massless field with two polarisation states.

Quantum numbers of the fermions in the electro-weak sector: The left-handed components (u_L, d_L) and (ν_L, e_L) transform as the fundamental representation under the $SU(2)$ group. The right-handed components u_R, d_R, ν_R and e_R transform as a singlet under the $SU(2)$ group.

In detail one has, where I_3 denotes the third component of the weak isospin, Y the hypercharge and Q the electric charge:

	I_3	Y	Q
u_L	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
d_L	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$
ν_L	$\frac{1}{2}$	-1	0
e_L	$-\frac{1}{2}$	-1	-1

	I_3	Y	Q
u_R	0	$\frac{4}{3}$	$\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
ν_R	0	0	0
e_R	0	-2	-1

The electric charge is given by the Gell-Mann-Nishijima formula:

$$Q = I_3 + \frac{Y}{2}$$

Remark: The table contains a right-handed neutrino, which does not interact with any other particle.

1.4.3 Spin 0 particles

The Higgs boson: Within the standard model there is a complex scalar field, transforming as the fundamental representation of $SU(2)$. This field is conventionally parametrised as follows:

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}.$$

$\phi^+(x)$ is a complex field (two real components). The three components $\phi^+(x)$ and $\chi(x)$ are absorbed as the longitudinal modes of $W_\mu^\pm$ and Z_μ. $H(x)$ is the Higgs field.

1.5 Experiments

The first experiments were fixed-target experiments (**deep inelastic scattering** of electrons on proton targets).

Accelarators:

- LEP, e^+e^-, 210 GeV, $L = 10^{32}$ cm^{-2} s^{-1};

- TEVATRON, $p\bar{p}$, 1.96 TeV, $L = 5 \cdot 10^{30}$ cm^{-2} s^{-1};

- HERA, $e^- p$, e^- : 30 GeV, p : 960 GeV, $L = 75 \cdot 10^{30}$ cm^{-2} s^{-1};

- LHC, pp, 14 TeV, $L = 10^{34}$ cm^{-2} s^{-1};

- Linear Collider (planned), $e^+ e^-$, 500 GeV, $L = 5 \cdot 10^{34}$ cm^{-2} s^{-1};

Quarks and gluons are not directly observed in these experiments. Instead one observes **hadronic jets**. A jet is a bunch of particles moving in the same direction. Particles in a jet are not necessarily elementary.

1.6 Observed, but not elementary particles

Due to confinement, quarks and gluons cannot be observed as free particles. In experiments we observe particles which are colour-singlets like **mesons** and **baryons**. Within the quark model, mesons are $q\bar{q}$-states and baryons qqq-states. Mesons and baryons are called collectively **hadrons**.
Examples are:

Mesons: Pions, kaons, η's, D-mesons, J/ψ, ...

Baryons: protons, neutrons, Σ, Ξ, ...

2 Review of quantum mechanics

2.1 The harmonic oscillator in classical mechanics

Let us start with classical mechanics and recall the Lagrange and Hamilton description of the non-relativistic harmonic oscillator in classical mechanics. The Lagrange function for the harmonic oscillator reads

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2.$$

From the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} = 0$$

follows the equation of motion

$$\ddot{x} + \omega^2 x^2 = 0,$$

which has the solution

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}.$$

Remark: The conjugate momentum is given by

$$p = \frac{\delta L}{\delta \dot{x}} = m\dot{x},$$

and the Hamilton function reads

$$H = p\dot{x} - L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

2.2 The harmonic oscillator in quantum mechanics

In quantum mechanics the harmonic oscillator is described by a wave function $\psi(x,t)$. This wave function can be expanded into an orthonormal basis. We denote by $|x;t_0\rangle_H$ a wave function in the Heisenberg picture, which is at the time $t = t_0$ an eigenvector of the Heisenberg position operator $\hat{x}_H(t)$ with eigenvalue x:

$$\hat{x}_H(t_0)\,|x;t_0\rangle_H = x\,|x;t_0\rangle_H.$$

In general, the position operator does not commute with the Hamilton operator, therefore for $t \neq t_0$ the state will in general not be an eigenstate of $\hat{x}_H(t)$. Furthermore we remark that a state in the Heisenberg picture is time-independent. The label t_0 refers to the time, where the state $|x;t_0\rangle$ is an eigenvector of the time-dependent position operator $\hat{x}_H(t)$. The relation between the states in the Heisenberg picture and in the Schrödinger picture is

$$|x;t_0\rangle_H = e^{i\hat{H}t}\,|x,t;t_0\rangle_S.$$

At time $t = t_0$ the Schrödinger state $|x, t_0; t_0\rangle$ is an eigenstate of the Schrödinger position operator $\hat{x}_S$:

$$\hat{x}_S \, |x, t_0; t_0\rangle_S \;\; = \;\; x \, |x, t_0; t_0\rangle_S.$$

We are interested in the transition amplitude

$${}_H\langle x_f; t_f | x_i; t_i\rangle_H,$$

which gives us the probability that the system which was in the eigenstate $|x_i; t_i\rangle_H$ at time t_i will be found in the state $|x_f; t_f\rangle_H$ at time t_f. We discuss the transition amplitude for pedagocical purposes: On the one hand, the transition amplitude can be worked out in non-relativistic quantum mechanics. On the other hand, the transition amplitude is already close to objects (scattering amplitudes), which we will study in quantum field theory. We will discuss two methods for the calculation of the transition amplitude: one method based on operators, the other on path integrals. Both methods have a generalisation to quantum field theory.

2.2.1 Operator formalism

Let us first compute the transition amplitude with the operator formalism. This is the standard method treated in most textbooks of quantum mechanics. Here, we present only the main formulae, but do not give any derivations.

The time evolution of the wave function in the Schrödinger picture is given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) \;\; = \;\; \hat{H} \psi(x,t),$$

where the Hamilton operator is given by

$$\hat{H} \;\; = \;\; \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2.$$

We make the ansatz

$$\psi(x,t) \;\; = \;\; \hat{U}(t, t_i) |x, t_i; t_i\rangle_S$$

The evolution operator satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_i) \;\; = \;\; \hat{H} \hat{U}(t, t_i).$$

If the Hamilton operator is time-independent (as it is for the harmonic oscillator), the solution for $\hat{U}(t, t_i)$ is given by

$$\hat{U}(t, t_i) \;\; = \;\; \exp\left(-\frac{i}{\hbar} (t - t_i) \cdot \hat{H} \right).$$

Remark: If the Hamilton operator $\hat{H}$ depends on the time t, a formal solution for $\hat{U}(t,t_i)$ is given by

$$\hat{U}(t,t_i) = T\exp\left(-\frac{i}{\hbar}\int\limits_{t_i}^{t} dt'\hat{H}(t')\right).$$

Here, T denotes the time-ordering operator, which orders operators from right to left in non-decreasing time. Expanding the exponential one obtains

$$\begin{aligned}\hat{U}(t,t_i) &= 1-\frac{i}{\hbar}\int\limits_{t_i}^{t} dt_1\hat{H}(t_1)+\left(\frac{i}{\hbar}\right)^2\int\limits_{t_i}^{t} dt_1\hat{H}(t_1)\int\limits_{t_i}^{t_1} dt_2\hat{H}(t_2) \\ &\quad -\left(\frac{i}{\hbar}\right)^3\int\limits_{t_i}^{t} dt_1\hat{H}(t_1)\int\limits_{t_i}^{t_1} dt_2\hat{H}(t_2)\int\limits_{t_i}^{t_2} dt_3\hat{H}(t_3)+....\end{aligned}$$

Note that the factor $1/n!$ disappears.

To determine $\hat{U}(t,t_i)|x,t_i;t_i\rangle_S$ we expand $|x,t_i;t_i\rangle_S$ into eigenstates of the Hamilton operator. These eigenstates will be labelled $|n\rangle$ and we have

$$\hat{H}|n\rangle = E_n|n\rangle.$$

Therefore

$$\hat{U}(t,t_i)|x,t_i;t_i\rangle_S = \exp\left(-\frac{i}{\hbar}(t-t_i)\cdot\hat{H}\right)|x,t_i;t_i\rangle_S = \sum_n e^{-\frac{i}{\hbar}(t-t_i)E_n}|n\rangle\langle n|x,t_i;t_i\rangle_S.$$

To find these eigenstates we define two operators

$$\hat{a} = \frac{\omega m\hat{x}+i\hat{p}}{\sqrt{2\omega m\hbar}}, \qquad \hat{a}^\dagger = \frac{\omega m\hat{x}-i\hat{p}}{\sqrt{2\omega m\hbar}}.$$

If we introduce the characteristic length

$$x_0 = \sqrt{\frac{\hbar}{\omega m}},$$

we can equally write them as

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}+x_0\frac{d}{dx}\right), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0}-x_0\frac{d}{dx}\right).$$

$\hat{a}$ is called lowering operator or annihilation operator, $\hat{a}^\dagger$ is called raising operator or creation operator. From

$$[\hat{x},\hat{p}] = \left[\hat{x},\frac{h}{i}\frac{d}{dx}\right] = i\hbar$$

it follows that

$$\left[\hat{a},\hat{a}^\dagger\right] \;=\; 1.$$

The Hamilton operator can be rewritten as

$$\hat{H} \;=\; \frac{1}{2}\hbar\omega\left(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\right)=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right).$$

We call

$$\hat{N} \;=\; \hat{a}^\dagger\hat{a}$$

the number operator and the problem of finding the energy eigenstates is reduced to the problem of finding the eigenstates of the number operator. We have

$$n\langle n|n\rangle=\langle n|\hat{N}|n\rangle=\langle n|\hat{a}^\dagger\hat{a}n\rangle=\langle\hat{a}n|\hat{a}n\rangle\geq 0.$$

Therefore $n\geq 0$ and the lowest energy state corresponds to $n=0$. Since the norm of $\hat{a}|0\rangle$ vanishes, we have

$$\begin{aligned}\hat{a}|0\rangle &= 0,\\ \left(\frac{d}{dx}+\frac{x}{x_0^2}\right)|0\rangle &= 0.\end{aligned}$$

A solution is given by

$$|0\rangle \;=\; \left(\sqrt{\pi}x_0\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right).$$

One easily shows that

- $\hat{a}^\dagger|n\rangle$ is an eigenstate with eigenvalue $n+1$.
- $\hat{a}|n\rangle$ is an eigenstate with eigenvalue $n-1$.

Therefore one finds

$$|n\rangle \;=\; \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle=\left(2^n n!\sqrt{\pi}x_0\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)H_n\left(\frac{x}{x_0}\right),$$

where $H_n(t)$ are the Hermite polynomials.

The corresponding energies are given by

$$E_n \;=\; \hbar\omega\left(n+\frac{1}{2}\right).$$

Finally, we get

$$_H\langle x_f;t_f|x_i;t_i\rangle_H \;=\; \sum_n {}_S\langle x_f,t_f;t_f|n\rangle e^{-\frac{i}{\hbar}(t_f-t_i)E_n}\langle n|x,t_i;t_i\rangle_S.$$

2.2.2 Path integrals

An alternative approach to determine the transition amplitude ${}_H\langle x_f;t_f|x_i;t_i\rangle_H$ divides the time interval (t_f-t_i) into $n+1$ small sub-intervals with time steps at

$$t_i, t_1, t_2, ..., t_n, t_f.$$

At each intermediate time step we insert a complete set of states

$$\int\limits_{-\infty}^{\infty} dx\, |x;t_j\rangle_H\, {}_H\langle x;t_j| \;\; = \;\; 1.$$

Therefore

$${}_H\langle x_f;t_f|x_i;t_i\rangle_H \;\; = \;\; \int\limits_{-\infty}^{\infty} dx_n ... \int\limits_{-\infty}^{\infty} dx_1\, {}_H\langle x_f;t_f|x_n;t_n\rangle_H\, {}_H\langle x_n;t_n|x_{n-1};t_{n-1}\rangle_H\, ...\, {}_H\langle x_1;t_1|x_i;t_i\rangle_H.$$

Let us study ${}_H\langle x_{j+1};t_{j+1}|x_j;t_j\rangle_H$. If the time interval $(t_{j+1}-t_j)$ is small, we have

$$\begin{aligned} {}_H\langle x_{j+1};t_{j+1}|x_j;t_j\rangle_H \;\; &= \;\; {}_S\langle x_{j+1},t_{j+1};t_{j+1}|e^{-\frac{i}{\hbar}(t_{j+1}-t_j)\hat{H}}|x_j,t_j;t_j\rangle_S \;\; = \;\; \langle x_{j+1}|e^{-\frac{i}{\hbar}(t_{j+1}-t_j)\hat{H}}|x_j\rangle \\ &\approx \;\; \langle x_{j+1}|1-\frac{i}{\hbar}(t_{j+1}-t_j)\hat{H}|x_j\rangle, \end{aligned}$$

where we denoted the eigenfunctions of $\hat{x}_S$ simply by $|x\rangle$. We have

$$\langle x_{j+1}|x_j\rangle \;\; = \;\; \delta(x_{j+1}-x_j) = \frac{1}{\hbar}\int\limits_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar}p_j\left(x_{j+1}-x_j\right)\right).$$

Here we used the integral representation of the Dirac delta distribution:

$$\delta(x-y) \;\; = \;\; \int\limits_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-y)}.$$

We then obtain

$$\begin{aligned} &\langle x_{j+1}|1-\frac{i}{\hbar}(t_{j+1}-t_j)\hat{H}|x_j\rangle \\ &= \;\; \langle x_{j+1}|x_j\rangle - \frac{i}{\hbar}(t_{j+1}-t_j)\left\langle x_{j+1}\left|-\frac{\hbar^2}{2m}\frac{d^2}{dx_j^2}+\frac{1}{2}m\omega^2 x_j^2\right|x_j\right\rangle \\ &= \;\; \frac{1}{\hbar}\int\limits_{-\infty}^{\infty} \frac{dp_j}{2\pi} \exp\left(\frac{i}{\hbar}p_j\left(x_{j+1}-x_j\right)\right)\left(1-\frac{i}{\hbar}(t_{j+1}-t_j)\left(\frac{p_j^2}{2m}+\frac{1}{2}m\omega^2 x_j^2\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\exp\left(\frac{i}{\hbar}p_j\left(x_{j+1}-x_j\right)\right)\left(1-\frac{i}{\hbar}(t_{j+1}-t_j)H(x_j,p_j)\right)\\
&\approx \frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\exp\left(\frac{i}{\hbar}p_j\left(x_{j+1}-x_j\right)-\frac{i}{\hbar}(t_{j+1}-t_j)H(x_j,p_j)\right)\\
&\approx \frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\exp\left(\frac{i}{\hbar}(t_{j+1}-t_j)\left(p_j\dot{x}_j-H(x_j,p_j)\right)\right).
\end{aligned}
$$

Note that $H(x_j,p_j)$ denotes the Hamilton function, not the Hamilton operator. (The Hamilton operator is denoted by $\hat{H}$.) Let us set

$$\Delta T = i(t_{j+1}-t_j).$$

With the help of

$$\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{-\frac{1}{2}\alpha y^2+wy} = \frac{1}{\sqrt{2\pi\alpha}}e^{\frac{w^2}{2\alpha}}$$

we may perform the integration over p_j:

$$\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\exp\left(\frac{\Delta T}{\hbar}\left(p_j\dot{x}_j-\frac{p_j^2}{2m}\right)\right) = \sqrt{\frac{\hbar m}{2\pi\Delta T}}\exp\left(\frac{\Delta T}{\hbar}\frac{1}{2}m\dot{x}_j^2\right).$$

Therefore

$$
\begin{aligned}
\langle x_{j+1}|1-\frac{i}{\hbar}(t_{j+1}-t_j)\hat{H}|x_j\rangle &\approx \sqrt{\frac{m}{2\pi\hbar\Delta T}}\exp\left(\frac{\Delta T}{\hbar}\left(\frac{1}{2}m\dot{x}_j^2-\frac{1}{2}m\omega^2x_j^2\right)\right)\\
&= \sqrt{\frac{m}{2\pi\hbar\Delta T}}\exp\left(\frac{\Delta T}{\hbar}L(x_j,\dot{x}_j)\right).
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
{}_H\langle x_f;t_f|x_i;t_i\rangle_H &= \int_{-\infty}^{\infty}dx_n\ldots\int_{-\infty}^{\infty}dx_1\;{}_H\langle x_f;t_f|x_n;t_n\rangle_H\;{}_H\langle x_n;t_n|x_{n-1};t_{n-1}\rangle_H\;\ldots\;{}_H\langle x_1;t_1|x_i;t_i\rangle_H\\
&= \left(\frac{m}{2\pi\hbar\Delta T}\right)^{\frac{n+1}{2}}\int_{-\infty}^{\infty}dx_n\ldots\int_{-\infty}^{\infty}dx_1\prod_{j=0}^{n}\exp\left(\frac{i}{\hbar}(t_{j+1}-t_j)L(x_j,\dot{x}_j)\right),
\end{aligned}
$$

with $t_0=t_i$. We rewrite this as

$${}_H\langle x_f,t_f|x_i,t_i\rangle_H \sim \int\mathcal{D}x(t)\exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f}dt\;L(x(t),\dot{x}(t))\right) = \int\mathcal{D}x(t)\exp\left(\frac{i}{\hbar}S\right).$$

Note the appearance of the Lagrange function $L(x(t),\dot{x}(t))$ and the action

$$S = \int_{t_i}^{t_f} dt\, L(x(t),\dot{x}(t)).$$

2.2.3 Summary

The quantum mechanical harmonic oscillator shows already several concepts, which will reappear later in quantum field theory. These are:

- Annihilation and creation operators.
- Transition amplitudes can be expressed as path integrals.
- The appearance of the Lagrange function and the action in the path integral.

3 Review of special relativity

3.1 Four-vectors and the metric

Four-vectors: The space-time coordinates (ct, x, y, z) are regarded as the components of a vector in a four-dimensional space.

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z.$$

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3), \\ &= (x^0, \vec{x}). \end{aligned}$$

Greek indices $\mu, \nu, \ldots$, which take the values $0, 1, 2, 3$, are used to denote the components of a four-vector. Latin indices $i, j, \ldots$ are used to denote the (spatial) components of a three-vector. They take the values $1, 2, 3$.

The distance between two points in four-dimensional space-time is

$$s_{ab} = (x_a^0 - x_b^0)^2 - (x_a^1 - x_b^1)^2 - (x_a^2 - x_b^2)^2 - (x_a^3 - x_b^3)^2.$$

$s_{ab}^2 > 0$ time-like distance;
there exists a frame, in which events a and b occur in the same place.

$s_{ab}^2 < 0$ space-like distance;
there exists a frame, in which events a and b occur at the same time.

$s_{ab}^2 = 0$ ligth-like distance;
light cone

Two events can only be related by causality, if the distance between them is ≥ 0. This follows directly from the finiteness of the speed of light.

We define the metric tensor $g_{\mu\nu}$ by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The distance is then given by

$$s_{ab} = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} \left(x_a^\mu - x_b^\mu \right) \left(x_a^\nu - x_b^\nu \right).$$

Summation convention of Einstein: The symbol of the sum is dropped and it is understood, that there is an implicit summation over any pair of indices, which occurs twice. Within a pait, one index has to be an upper index, the other one a lower index. Therefore:

$$s_{ab} = g_{\mu\nu}(x_a - x_b)^\mu (x_a - x_b)^\nu.$$

We call a four-vector x^μ with an upper index a contravariant four-vector, and we call a four-vector x_μ with a lower index a covariant four-vector. The relation between the two is given by

$$x_\mu = g_{\mu\nu} x^\nu.$$

Therefore we can write the distance equally as

$$s_{ab} = (x_a - x_b)_\mu (x_a - x_b)^\mu = (x_a - x_b)^\mu (x_a - x_b)_\mu.$$

Remark: The geometry defined by the quadratic form $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ is non-Euclidean. The special case of a four-dimensional space with metric $\text{diag}(1,-1,-1,-1)$ is often called Minkowski space.

3.2 The Lorentz group

Axioms for a group: Let G be a non-empty set with a binary operation. G is called a group, if it satisfies the axioms

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Existence of a neutral element: $e \cdot a = a$.
- Existence of an inverse: $a^{-1} \cdot a = e$.

Definition of the Lorentz group: Matrix group, which leaves the metric tensor $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ invariant:

$$\Lambda^T g \Lambda = g,$$

The same equation with indices:

$$\Lambda^\mu{}_\sigma g_{\mu\nu} \Lambda^\nu{}_\tau = g_{\sigma\tau}.$$

This group is denoted $O(1,3)$. It is easy to show that

$$(\det \Lambda)^2 = 1,$$

and therefore

$$\det \Lambda = \pm 1.$$

If we have in addition $\det \Lambda = 1$ the corresponding group is called the "proper" Lorentz group and denoted $SO(1,3)$.
One further distinguishes the cases whether the time direction is conserved or reversed. If

$$\Lambda^0{}_0 \;\; \geq \;\; 1,$$

the time direction is conserved and the corresponding group is called the orthochronous Lorentz group. If on the other hand

$$\Lambda^0{}_0 \;\; \leq \;\; -1,$$

then the time direction is reversed. Remark:

$$\left|\Lambda^0{}_0\right| \;\; \geq \;\; 1$$

follows from $\Lambda^\mu{}_\sigma g_{\mu\nu} \Lambda^\nu{}_\tau = g_{\sigma\tau}$ for $\sigma = \tau = 0$:

$$\left(\Lambda^0{}_0\right)^2 - \sum\limits_{j=1}^{3} \left(\Lambda^j{}_0\right)^2 \;\; = \;\; 1.$$

To summarise: The Lorentz group consists of four components, depending on which values the quantities

$$\det \Lambda \quad \text{and} \quad \Lambda^0{}_0$$

take. The "proper orthochronous Lorentz group" is defined by

$$\Lambda^\mu{}_\sigma g_{\mu\nu} \Lambda^\nu{}_\tau \;\; = \;\; g_{\sigma\tau}, \;\; \det \Lambda = 1, \;\; \Lambda^0{}_0 \geq 1,$$

and contains the identity. Elements of the group correspond to rotations in four-dimensional Minkowski space. Each rotation can be decomposed into rotations in the planes xy, yz, zx, tx, ty and tz. A spatial rotation in the xy-plane is given by

$$\Lambda^\mu{}_\nu \;\; = \;\; \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

and similar for the yz- and zx-planes. A boost in the tx-plane is given by

$$\Lambda^\mu{}_\nu \;\; = \;\; \left(\begin{array}{cccc} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

with

$$\sinh\phi = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \beta\gamma, \qquad \cosh\phi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma,$$

where we used the standard abreviations

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Elements of the other three components can be obtained from an element of the proper orthochronous Lorentz group and a discrete transformation of time reversal

$$\Lambda^{\mu}{}_{\nu} = \left(\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

and / or spatial inversion

$$\Lambda^{\mu}{}_{\nu} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right).$$

Tensors: $T^{\mu_1\mu_2...\mu_r}$ is called a tensor if it transforms under Lorentz transformations as

$$T'^{\mu_1\mu_2...\mu_r} = \Lambda^{\mu_1}{}_{\nu_1}\Lambda^{\mu_2}{}_{\nu_2}...\Lambda^{\mu_r}{}_{\nu_r}T^{\nu_1\nu_2...\nu_r}.$$

The number r is called the rank of the tensor.

Pseudo-tensors: Pseudo-tensors transform under elements of the proper orthochronous Lorentz group as tensors. Under the discrete transformations of time reversal and spatial inversion there is however an additional minus sign. Pseudo-tensors of rank zero are called pseudo-scalars, pseudo-tensors of rank one are called axial vectors.

Examples:

Rank 1: Positon vector x^μ, momentum vector p^μ.

Rank 2: Metric tensor $g^{\mu\nu}$.

Rank 4: Total anti-symmetric tensor (Levi-Civita tensor) $\varepsilon^{\mu\nu\rho\sigma}$. The total anti-symmetric tensor is defined by

$$\begin{aligned} \varepsilon_{0123} &= 1, \\ \varepsilon_{\mu\nu\rho\sigma} &= 1 \quad \text{if } (\mu,\nu,\rho,\sigma) \text{ is an even permutation of } (0,1,2,3), \\ \varepsilon_{\mu\nu\rho\sigma} &= -1 \quad \text{if } (\mu,\nu,\rho,\sigma) \text{ is an odd permutation of } (0,1,2,3), \\ \varepsilon_{\mu\nu\rho\sigma} &= 0 \quad \text{otherwise.} \end{aligned}$$

The total anti-symmetric tensor is a pseudo-tensor, its components are unchanged under time reversal and spatial inversion.

3.3 The Poincaré group

The Poincaré group consists of elements of the Lorentz group and translations. The group elements act on four vectors according to the following transformation law :

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}.$$

Λ describes rotations in four dimensional space-time (e.g. ordinary rotations on the spacial components plus boosts) whereas a describes translations.
The group multiplication law is given by

$$\{a_1, \Lambda_1\}\{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2\}.$$

The generators of the Poincaré group can be realised as differential operators :

$$\begin{aligned} P_{\mu} &= i\partial_{\mu}, \\ M_{\mu\nu} &= i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right). \end{aligned}$$

The algebra of the Poincaré group is given by

$$\begin{aligned} \left[M_{\mu\nu}, M_{\rho\sigma}\right] &= -i\left(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}\right), \\ \left[M_{\mu\nu}, P_{\sigma}\right] &= i\left(g_{\nu\sigma}P_{\mu} - g_{\mu\sigma}P_{\nu}\right), \\ \left[P_{\mu}, P_{\nu}\right] &= 0. \end{aligned}$$

The Poincaré algebra is a Lie algebra, but it is not semi-simple, since it has an Abelian non-trivial ideal (P_{μ}).
Casimir operators are M^2 and W^2 where

$$M^2 = P_{\mu}P^{\mu}, \qquad W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_{\nu}M_{\rho\sigma}.$$

W^{μ} is called the Lubanski-Pauli vector.

4 Review of classical field theory

4.1 The action principle

From classical mechanics we are familiar with Hamilton's principle of the least action for systems with a finite number of degrees of freedom. This principle generalises to systems with infinite many degrees of freedom:

- Let us start with a countable number of degrees of freedom. We will label the coordinates by q_i, where $i \in \mathbb{Z}$. As an example we consider a model of an elastic rod, described by points of mass m, separated at rest by a distance a and connected by massless springs with spring constant k. The kinetic and potential energies are

$$\begin{aligned} T &= \frac{1}{2}\sum_i m\dot{q}_i^2, \\ V &= \frac{1}{2}\sum_i k\left(q_{i+1}-q_i\right)^2. \end{aligned}$$

  The Lagrange function is then given by

$$L = T - V = \frac{1}{2}\sum_i \left(m\dot{q}_i^2 - k\left(q_{i+1}-q_i\right)^2\right) = \frac{1}{2}\sum_i a\left(\frac{m}{a}\dot{q}_i^2 - ka\left(\frac{q_{i+1}-q_i}{a}\right)^2\right).$$

  In the last expression we already arranged the terms in such a way that it will be easy to take the continuum limit. The action is a usual

$$S\left[\vec{q}(t)\right] = \int\limits_{t_1}^{t_2} dt\, L\left(\vec{q}(t), \dot{\vec{q}}(t)\right),$$

  where $\vec{q}(t)$ is an infinite-dimensional vector. The equations of motion follow as usual from

$$\delta S = 0$$

  and lead to the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = 0,$$

  for $i \in \mathbb{Z}$.

- Let us now consider a non-countable number of degrees of freedom. Instead of the integer i we will label them by x. Furthermore it will be convenient to change the notation from q_x to $q(x)$ and from $q_x(t)$ to $q(t,x)$. In the example above we take the limit $a \to 0$ and we set

$$\lim_{a\to 0}\frac{m}{a} = \mu, \qquad \lim_{a\to 0} ka = Y.$$

Furthermore the expression $(q_{i+1} - q_i)/a$ becomes in the continuum limit

$$\lim_{a \to 0} \frac{q(t,x+a) - q(t,x)}{a} = \frac{\partial q(t,x)}{\partial x}.$$

We obtain for the Lagrange function

$$L = \frac{1}{2} \int\limits_{-\infty}^{\infty} dx \left[\mu \left(\frac{\partial q(t,x)}{\partial t} \right)^2 - Y \left(\frac{\partial q(t,x)}{\partial x} \right)^2 \right].$$

The expression

$$\mathcal{L} = \frac{1}{2} \left[\mu \left(\frac{\partial q(t,x)}{\partial t} \right)^2 - Y \left(\frac{\partial q(t,x)}{\partial x} \right)^2 \right]$$

is called the **Lagrange density** or **Lagrangian**. The action is given by

$$S = \int\limits_{t_1}^{t_2} dt \int\limits_{-\infty}^{\infty} dx \, \mathcal{L}.$$

We may view $q(t,x)$ as a classical field in a $(1+1)$-dimensional space-time. Note that the Lagrange density is a function of the time derivative $\partial q / \partial t$ and the spatial derivative $\partial q / \partial x$.

We now consider the generalisation to four-dimensional space-time. We assume that the Lagrange density is a function of the field $\psi(x)$ and its first derivative $\partial_\mu \psi(x)$:

$$\mathcal{L} \left(\psi(x), \partial_\mu \psi(x) \right).$$

The motivation is as follows: Classical mechanics suggests to include a dependence on $\psi(x)$ and the time derivative $\partial_0 \psi(x)$. Lorentz invariance instructs us not to single out a specific direction in space-time. We therefore allow a dependence on all first derivatives $\partial_\mu \psi(x)$. In principle we could include also second or higher derivatives, however for our purpose it will be sufficient to restrict us to first derivatives. The action is given by

$$S = \int\limits_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} = \frac{1}{c} \int d^4x \, \mathcal{L}.$$

From the principle of least action

$$\delta S = 0,$$

we can derive the Euler-Lagrange equations as in classical mechanics, if we assume that the variation of the field vanishes on two hyper-surfaces $t = t_1$ and $t = t_2$:

$$\begin{aligned}
\delta S &= \int\limits_{t_1}^{t_2} dt \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi(x)\right)} \delta\left(\partial_\mu \psi(x)\right) \right] \\
&= \int\limits_{t_1}^{t_2} dt \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi(x)\right)} \partial_\mu \left(\delta\psi(x)\right) \right] \\
&= \int\limits_{t_1}^{t_2} dt \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi(x)\right)} \right] \delta\psi.
\end{aligned}$$

Therefore

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \psi(x)\right)} = 0.$$

This equation is also called the field equation.

4.2 Examples of classical fields

In the following we will review the most important cases of fields: These are the Klein-Gordon field, describing particles of spin 0, the Dirac field, describing particles of spin $1/2$ and the Maxwell field, describing particles of spin 1. We review the properties, when these fields are treated as classical fields, i.e. fields which satisfy a specific partial differential equation, called field equation. In the case of spin 0, the field equation is the Klein-Gordon equation, in the case of spin $1/2$ the field equation is the Dirac equation and in the case of spin 1 the field equations are the (inhomogeneous) Maxwell equations.

4.2.1 The Klein-Gordon field

The Lagrange density for a real scalar field:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2.$$

The corresponding Euler-Lagrange equation yields the Klein-Gordon equation:

$$\left(\Box + m^2\right) \phi = 0.$$

The Lagrange density for a complex scalar field reads

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = \left(\partial_\mu \phi^*\right) \left(\partial^\mu \phi\right) - m^2 \phi^* \phi.$$

In the complex case we treat ϕ and ϕ^* as two independent fields. Variation with respect to ϕ^* yields

$$\left(\Box + m^2\right)\phi = 0,$$

variation with respect to ϕ gives

$$\left(\Box + m^2\right)\phi^* = 0.$$

4.2.2 The Dirac field

Although Dirac spinors are associated with the quantum mechanical description of spin, we may ignore this fact for the moment and simply take a Dirac spinor as a four-component vector with complex entries. A Dirac field associates to every space-time point x a four-component spinor $\psi_\alpha(x)$, where α takes the values $\alpha \in \{1,2,3,4\}$. Note that α is not a Lorentz index, it is a Dirac index. (We use greek letters from the beginning of the greek alphabet for spinor indices and greek letters from the middle of the greek alphabet for Lorentz indices.)

Let us define the Dirac matrices. These are (4×4)-matrices, which satisfy the anti-commutation rules

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}.$$

(The anti-commutator of two matrices is $\{A,B\} = AB + BA$.) In addition, there is a fifth matrix γ_5, defined by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$

which satisfies

$$\{\gamma^\mu, \gamma_5\} = 0.$$

There are various representations for the Dirac matrices, a convenient choice is the Weyl representation. In order to define the Weyl representation we first recall the Pauli matrices. These are 2×2-matrices, given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We write $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Next, we define the 4-dimensional σ^μ-matrices (and $\bar{\sigma}^\mu$-matrices):

$$\sigma^\mu_{A\dot{B}} = (1, -\vec{\sigma}), \qquad \bar{\sigma}^{\mu\dot{A}B} = (1, \vec{\sigma}).$$

Here 1 denotes the 2×2-identity matrix. There are four 2×2-matrices $\sigma^0_{A\dot{B}}$, $\sigma^1_{A\dot{B}}$, $\sigma^2_{A\dot{B}}$ and $\sigma^3_{A\dot{B}}$. The indices A and $\dot{B}$ take values $A, \dot{B} \in \{1,2\}$. Analog statements hold for $\bar{\sigma}^{\mu\dot{A}B}$. We are now in a position to give the Weyl representation for the Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Each entry is a 2×2-matrix.

The Lagrange density for the Dirac field depends on four-component spinors $\psi_\alpha(x)$ and the Dirac adjoint $\bar{\psi}_\alpha(x) = \left(\psi^\dagger(x)\gamma^0\right)_\alpha$:

$$\mathcal{L}(\psi, \bar{\psi}, \partial_\mu \psi) = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi.$$

The Euler-Lagrange equations yield the Dirac equations

$$\begin{aligned}\left(i\gamma^\mu \partial_\mu - m\right)\psi &= 0,\\ \bar{\psi}\left(i\gamma^\mu \overleftarrow{\partial}_\mu + m\right) &= 0.\end{aligned}$$

The arrow on top of the derivative indicates that the derivative acts to the left.

4.2.3 The Maxwell field

As our final example we consider the electromagnetic field, described by the gauge potential $A_\mu(x)$. The Lagrange density is given by

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Half of Maxwell´s equations follow from the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta A_\mu} - \partial_\nu \frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu\right)} = 0.$$

This yields

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= 0,\\ \text{or equivalently}\quad \Box A^\mu - \partial^\mu \partial_\nu A^\nu &= 0.\end{aligned}$$

Remark: The first set of Maxwell's equations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

is fulfilled identically with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Gauge invariance: If $A^\mu = (\phi, \vec{A})$, then the electric and magnetic fields are given by

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A},\\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}.\end{aligned}$$

Therefore, the potential A_μ determines the electric and magnetic fields. We may ask the reverse question: Given $\vec{E}$ and $\vec{B}$, does this define uniquely the potential A_μ ? This is not the case. We may add to A_μ the divergence of an arbitrary function:

$$A_\mu(x) \quad \to A_\mu(x) + \partial_\mu \Lambda(x).$$

Since

$$\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda \quad = \quad 0,$$

this leaves the field stregth

$$F^{\mu\nu} \quad = \quad \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}$$

and hence the electric and magnetic fields unchanged. Therefore we may impose additional conditions on the gauge potential. A common choice is the covariant Lorenz gauge

$$\partial_\mu A^\mu \quad = \quad 0.$$

A variational problem in the presence of constraints is solved with Lagrange multipliers. This leads us to the Lagrange density

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) \quad = \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2$$

and the equation of motion

$$\begin{aligned} \Box A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\nu A^\nu \quad &= \quad 0, \\ \left[\Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu\right] A^\nu \quad &= \quad 0. \end{aligned}$$

5 Quantum field theory: The canonical formalism

There are several approaches to quantum field theory. In this course we will focus on two of them: The canonical formalism and the path integral formalism. We start with the canonical formalism.

Within the **canonical approach** a system is described by a **state vector** and **operators** acting on the state vectors. An important operator will be the **Hamilton operator** (or **Hamiltonian**). The Hamilton operator can be obtained by analogy with classical field theory. In classical field theory we have a Hamilton density, consisting of the fields and derivatives of the fields. Promoting the fields to operators gives us the Hamilton operator of quantum field theory.

Within the **Schrödinger picture**, the time evolution of a state is governed by the Hamilton operator. We will also consider the **Heisenberg picture**, where the time-dependence is carried by the operators, while the states are time-independent. Finally, a third picture will be useful for practical calculations: the **interaction picture**.

Remark: Classical physics is concerned with classical point-like particles and classical fields. In quantum mechanics we describe particles by a wave function. This is often called "first quantisation". If fields are present, they are treated classically. We would like to treat (matter) particles and (force) fields, which also have a particle nature, on equal footing. This is achieved in quantum field theory, where fields are described by operators. This is often called "second quantisation". However we should add a warning here: The expression "second quantisation" might give wrong allusions. Also in quantum field theory we quantise only once. In fact we will soon see that we may treat a quantum field as a collection of infinite many harmonic oscillators. Each harmonic oscillator is quantised as in quantum mechanics.

5.1 The Klein-Gordon field as harmonic oscillators

We start with the simplest example, a real scalar field as a classical field with Lagrange density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2.$$

We define the **canonical conjugated momentum field** by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}.$$

For the Klein-Gordon field we find

$$\pi(x) = \dot{\phi}(x).$$

The Hamilton function is given by

$$H = \int d^3x \left[\pi(x)\dot{\phi}(x) - \mathcal{L}\right] = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2\right].$$

We write

$$H = \int d^3x\, \mathcal{H}, \quad \mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2.$$

$\mathcal{H}$ is called the **Hamiltonian**. The energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} = (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}\left(\partial_\rho\phi\right)(\partial^\rho\phi) + \frac{1}{2}g^{\mu\nu}m^2\phi^2$$

If the Lagrange density does not depend explicitly on x, i.e. the x-dependence is only through $\phi(x)$, then Noether's theorem implies

$$\partial_\mu T^{\mu\nu} = 0.$$

The four "conserved charges" are

$$H = \int d^3x\, T^{00} = \int d^3x\, \mathcal{H}$$

and

$$P^i = \int d^3x\, T^{0i} = -\int d^3x\, \pi\partial_i\phi.$$

The quantity

$$-\pi\vec{\nabla}\phi$$

is called the **momentum density** and

$$\vec{P} = -\int d^3x\, \pi\vec{\nabla}\phi$$

the **total momentum of the classical field**.

Please note the difference between the canonical conjugated momentum field $\pi(x)$ and the momentum density $[-\pi(x)\vec{\nabla}\phi(x)]$. These are not equal. This can already be seen by the fact that the former is a scalar quantity, wheras the latter is vector-valued.

Let us write the classical Klein-Gordon field as a Fourier integral with respect to $\vec{x}$:

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}}\phi(t,\vec{p}).$$

Then the Klein-Gordon equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\vec{p}|^2 + m^2\right)\right]\phi(t,\vec{p}) = 0.$$

This is the same as the equation of a harmonic oscillator with frequency

$$\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}.$$

For a harmonic oscillator we know how to make the transition from the classical picture to the quantum world. The Hamilton operator for a quantum-mechanical harmonic osciallator with this frequency is given by

$$\hat{H} = \frac{1}{2}\hat{\pi}_{\vec{p}}^2 + \frac{1}{2}\omega_{\vec{p}}^2\hat{\phi}_{\vec{p}}^2.$$

The corresponding creation and annihilation operators are

$$\hat{a}_{\vec{p}} = \sqrt{\frac{\omega_{\vec{p}}}{2}}\hat{\phi}_{\vec{p}} + \frac{i}{\sqrt{2\omega_{\vec{p}}}}\hat{\pi}_{\vec{p}}, \qquad \hat{a}_{\vec{p}}^\dagger = \sqrt{\frac{\omega_{\vec{p}}}{2}}\hat{\phi}_{\vec{p}} - \frac{i}{\sqrt{2\omega_{\vec{p}}}}\hat{\pi}_{\vec{p}}.$$

We may solve these equations for $\hat{\phi}_{\vec{p}}$ and $\hat{\pi}_{\vec{p}}$ and obtain

$$\hat{\phi}_{\vec{p}} = \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}} + \hat{a}_{\vec{p}}^\dagger\right), \qquad \hat{\pi}_{\vec{p}} = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(\hat{a}_{\vec{p}} - \hat{a}_{\vec{p}}^\dagger\right).$$

Note that the operators $\hat{\phi}_{\vec{p}}$ and $\hat{\pi}_{\vec{p}}$ correspond to a single momentum mode $\vec{p}$.

5.2 The Schrödinger picture

We are now in a position to present a quantum field as an operator. The basic idea is to take a linear superposition of momentum modes, where each individual momentum mode behaves like a quantum mechanical harmonic oscillator.

We start with the Schrödinger picture, where the operators do not depend on time. In the Schrödinger picture we denote operators by $\hat{O}(\vec{x})$, or $\hat{O}_S(\vec{x})$, if we want to emphasize that an operator refers to the Schrödinger picture. In the Schrödinger picture, operators may depend on the spatial coordinates $\vec{x}$, but not on the time coordinate t.

In the Schrödinger picture the time dependence is carried by the states. States are denoted by $|X\rangle$. If we want to emphasize that the states carry the time-dependence we will write $|X,t\rangle$. If we further want to emphasize that a state refers to the Schrödinger picture we will put a subscript S and write $|X,t\rangle_S$.

We start from

$$\begin{aligned} \left[\hat{q}_i, \hat{p}_j\right] &= i\delta_{ij}, \\ \left[\hat{q}_i, \hat{q}_j\right] &= \left[\hat{p}_i, \hat{p}_j\right] = 0, \end{aligned}$$

which for a continous system becomes

$$\begin{aligned} \left[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})\right] &= i\delta^3(\vec{x}-\vec{y}), \\ \left[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})\right] &= \left[\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})\right] = 0. \end{aligned}$$

These are called the **canonical commutation relations**. We write a Fourier representation for the fields:

$$\begin{aligned}\hat{\phi}(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+\hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right),\\ \hat{\pi}(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(\hat{a}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}-\hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right).\end{aligned}$$

Note that we are considering a real field. Therefore we would like to have that $\hat{\phi}(\vec{x})$ is self-adjoint:

$$\left(\hat{\phi}(\vec{x})\right)^\dagger = \hat{\phi}(\vec{x}).$$

In addition we would like to have

$$\left(\hat{a}^\dagger\right)^\dagger = \hat{a}.$$

These two conditions explain the sign in the exponent $e^{-i\vec{p}\cdot\vec{x}}$ in the term proportional to $\hat{a}^\dagger_{\vec{p}}$ in the Fourier representation of $\hat{\phi}(\vec{x})$. The inverse formulae read

$$\begin{aligned}\hat{a}_{\vec{p}} &= \int d^3x\left(\sqrt{\frac{\omega_{\vec{p}}}{2}}\hat{\phi}(\vec{x})+\frac{i}{\sqrt{2\omega_{\vec{p}}}}\hat{\pi}(\vec{x})\right)e^{-i\vec{p}\cdot\vec{x}},\\ \hat{a}^\dagger_{\vec{p}} &= \int d^3x\left(\sqrt{\frac{\omega_{\vec{p}}}{2}}\hat{\phi}(\vec{x})-\frac{i}{\sqrt{2\omega_{\vec{p}}}}\hat{\pi}(\vec{x})\right)e^{i\vec{p}\cdot\vec{x}}.\end{aligned}$$

The commutation relation becomes

$$\left[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}\right] = (2\pi)^3\delta^3(\vec{p}-\vec{q}).$$

In detail:

$$\begin{aligned}\left[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}\right] &= \left[\int d^3x\left(\sqrt{\frac{\omega_{\vec{p}}}{2}}\hat{\phi}(\vec{x})+\frac{i}{\sqrt{2\omega_{\vec{p}}}}\hat{\pi}(\vec{x})\right)e^{-i\vec{p}\cdot\vec{x}},\int d^3y\left(\sqrt{\frac{\omega_{\vec{q}}}{2}}\hat{\phi}(\vec{y})-\frac{i}{\sqrt{2\omega_{\vec{q}}}}\hat{\pi}(\vec{y})\right)e^{i\vec{q}\cdot\vec{y}}\right]\\ &= \int d^3x\int d^3y e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{q}\cdot\vec{y}}\left(-\frac{i}{2}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\left[\hat{\phi}(\vec{x}),\hat{\pi}(\vec{y})\right]+\frac{i}{2}\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\left[\hat{\pi}(\vec{x}),\hat{\phi}(\vec{y})\right]\right)\\ &= \int d^3x\int d^3y e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{q}\cdot\vec{y}}\left(\frac{1}{2}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}+\frac{1}{2}\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\right)\delta^3(\vec{x}-\vec{y})\\ &= \frac{1}{2}\left(\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}+\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\right)\int d^3x e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\\ &= \frac{1}{2}\left(\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}+\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}\right)(2\pi)^3\delta^3(\vec{p}-\vec{q}) = (2\pi)^3\delta^3(\vec{p}-\vec{q}).\end{aligned}$$

The remaining commutation relations for the creation and annihilation operators are

$$\left[\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}\right] = \left[\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\right] = 0.$$

Let us summarise: The equivalence of the canonical commutation relations

$$\left[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})\right] = i\delta^3(\vec{x}-\vec{y}), \qquad \left[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})\right] = \left[\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})\right] = 0$$

in momentum space are the relations

$$\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\right] = (2\pi)^3\delta^3(\vec{p}-\vec{q}), \qquad \left[\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}\right] = \left[\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\right] = 0.$$

The Hamiltonian becomes

$$\hat{H} = \int d^3x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\vec{\nabla}\hat{\phi}\right)^2 + \frac{1}{2}m^2\hat{\phi}^2\right] = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{1}{2}\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}\right]\right)$$

The second term is proportional to $\delta^3(\vec{0})$ and gives an infinite constant. Such a term can be expected: A single harmonic oscillator has the ground state energy $\frac{1}{2}\omega$, summing over an infinite number of harmonic oscillators yields an infinite ground state energy. As experiments can only measure energy differences from the ground state, we will ignore this term. Expressing the Hamiltonian in terms of annihilation and creation operators we will encounter at intermediate stages expressions proportional to $\hat{a}_{\vec{p}}\hat{a}_{\vec{p}}$ and $\hat{a}^\dagger_{\vec{p}}\hat{a}^\dagger_{\vec{p}}$. These expressions are in addition proportional to $\omega^2_{\vec{p}} - |\vec{p}|^2 - m^2 = 0$ and vanish therefore.

The commutation relations of the Hamiltonian with the annihilation and creation operators are

$$\left[\hat{H}, \hat{a}^\dagger_{\vec{q}}\right] = \omega_{\vec{q}}\hat{a}^\dagger_{\vec{q}}, \qquad \left[\hat{H}, \hat{a}_{\vec{q}}\right] = -\omega_{\vec{q}}\hat{a}_{\vec{q}}.$$

Let us look at the total momentum operator

$$\hat{\vec{P}} = -\int d^3x\, \hat{\pi}(\vec{x})\vec{\nabla}\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{1}{2}\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{p}}\right]\right).$$

Again, the second term will give a contribution proportional to $\delta^3(\vec{0})$, which we will ignore. In order to show the equality for the last equal sign in the equation above, we will encounter the following integrals:

$$\int \frac{d^3p}{(2\pi)^3}\,\vec{p}\, a_{\vec{p}}\, a_{-\vec{p}} \quad \text{and} \quad \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\, a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}.$$

The integrands of these integrals are anti-symmetric under $\vec{p} \to -\vec{p}$. Therefore these integrals vanish. With the same reasoning one can argue that $\vec{p}\delta^3(\vec{0})$ is anti-symmetric under $\vec{p} \to -\vec{p}$ and therefore the infinite constant has to vanish after integration over d^3p.

Let us look at the commutation relations of $\hat{\vec{P}}$ with $\hat{a}^\dagger_{\vec{q}}$ and $\hat{a}_{\vec{q}}$. We have

$$\begin{aligned}\left[\hat{\vec{P}},\hat{a}^\dagger_{\vec{q}}\right] &= \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{q}} - \hat{a}^\dagger_{\vec{q}}\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\right) = \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\hat{a}^\dagger_{\vec{p}}\left[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\,\vec{p}\hat{a}^\dagger_{\vec{p}}(2\pi)^3\,\delta^3\left(\vec{p}-\vec{q}\right) = \vec{q}\hat{a}^\dagger_{\vec{q}}.\end{aligned}$$

A similar relation (with an additional minus sign) holds for $\hat{a}_{\vec{q}}$. Thus we have

$$\left[\hat{\vec{P}},\hat{a}^\dagger_{\vec{q}}\right] = \vec{q}\hat{a}^\dagger_{\vec{q}}, \qquad \left[\hat{\vec{P}},\hat{a}_{\vec{q}}\right] = -\vec{q}\hat{a}_{\vec{q}}.$$

We may combine the Hamilton operator $\hat{H}$ and the three-momentum operator $\hat{\vec{P}}$ into a four-momentum operator $\hat{P}^\mu$:

$$\hat{P}^\mu = \left(\hat{H},\hat{\vec{P}}\right) = \int \frac{d^3p}{(2\pi)^3}\,p^\mu\left(\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \frac{1}{2}\left[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{p}}\right]\right),$$

with $p^\mu = (\omega_{\vec{p}},\vec{p})$ and $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2+m^2}$. From now on we will write

$$E_{\vec{p}} = \omega_{\vec{p}} = +\sqrt{|\vec{p}|^2+m^2}$$

Note that the energy is always positive. We have

$$\left[\hat{P}^\mu,\hat{a}^\dagger_{\vec{q}}\right] = q^\mu\hat{a}^\dagger_{\vec{q}}, \qquad \left[\hat{P}^\mu,\hat{a}_{\vec{q}}\right] = -q^\mu\hat{a}_{\vec{q}}.$$

Let us now discuss the states. The **ground state** is defined as the state, which is annihilated by all annihilation operators. We denote the ground state by $|0\rangle$. Thus

$$\hat{a}_{\vec{p}}\,|0\rangle = 0 \text{ for all } \hat{a}_{\vec{p}}.$$

If we drop the infinite constant above, the ground state has energy $E=0$ and momentum $\vec{p}=0$. All other states can be obtained by acting on $|0\rangle$ with creation operators. As a first example let us consider the state $\hat{a}^\dagger_{\vec{p}}|0\rangle$. We have

$$\hat{P}^\mu\left(\hat{a}^\dagger_{\vec{q}}|0\rangle\right) = \left(\hat{a}^\dagger_{\vec{q}}\hat{P}^\mu + \left[\hat{P}^\mu,\hat{a}^\dagger_{\vec{q}}\right]\right)|0\rangle = q^\mu\left(\hat{a}^\dagger_{\vec{q}}|0\rangle\right).$$

Thus, $\hat{a}^\dagger_{\vec{p}}|0\rangle$ is an eigenstate of $\hat{P}^\mu$ with momentum $\vec{p}$ and energy $E_{\vec{p}} = \sqrt{|\vec{p}|^2+m^2}$. The state $\hat{a}^\dagger_{\vec{p}}|0\rangle$ is called a **one-particle state**.

Next, consider the state

$$\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}|0\rangle.$$

This state is again an eigenstate of $\hat{P}^\mu$ with momentum $\vec{p}_1 + \vec{p}_2$ and energy $E_{\vec{p}_1} + E_{\vec{p}_2}$. The state $\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}|0\rangle$ is called a **two-particle state**. Further note that

$$\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}|0\rangle = \hat{a}^\dagger_{\vec{p}_2}\hat{a}^\dagger_{\vec{p}_1}|0\rangle,$$

since $\hat{a}^\dagger_{\vec{p}_1}$ and $\hat{a}^\dagger_{\vec{p}_2}$ commute.

Following this pattern we can construct ***n*-particle states** as

$$\hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}...\hat{a}^\dagger_{\vec{p}_n}|0\rangle$$

and therefore the full spectrum of the Hamilton operator.

Summary: The spectrum of the Hamilton operator

$$\hat{H} = = \int d^3x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\vec{\nabla}\hat{\phi}\right)^2 + \frac{1}{2}m^2\hat{\phi}^2\right]$$

is obtained from the ground state $|0\rangle$ by successively applying creation operators $\hat{a}^\dagger_{\vec{p}}$. All states are reached this way.

Normalisation: For the ground state we choose the normalisation

$$\langle 0|0\rangle = 1.$$

For one-particle state we choose the normalisation

$$\langle \vec{p}|\vec{q}\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3\left(\vec{p}-\vec{q}\right).$$

Therefore

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\hat{a}^\dagger_{\vec{p}}|0\rangle.$$

Remark: This normalisation is Lorentz invariant. Consider the boost

$$\begin{aligned} E' &= \gamma E - \beta\gamma p_3, \\ p_3' &= -\beta\gamma E + \gamma p_3. \end{aligned}$$

From

$$\delta\left(f(x) - f(x_0)\right) = \frac{1}{|f'(x_0)|}\delta\left(x - x_0\right)$$

it follows that

$$\begin{aligned} \delta^3(\vec{p}-\vec{q}) &= \delta^3(\vec{p}'-\vec{q}')\cdot\frac{dp_3'}{dp_3} = \delta^3(\vec{p}'-\vec{q}')\gamma\left(1-\beta\frac{dE}{dp_3}\right) = \delta^3(\vec{p}'-\vec{q}')\gamma\left(1-\beta\frac{p_3}{E}\right) \\ &= \delta^3(\vec{p}'-\vec{q}')\frac{E'}{E}. \end{aligned}$$

Remark:

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} = \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2 - m^2)\theta(p_0)$$

is a Lorentz-invariant 3-momentum integral. Therefore, if $f(p)$ is Lorentz-invariant, so is

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} f(p).$$

The normalisation of the n-particle state is

$$|\vec{p}_1, \vec{p}_2, ..., \vec{p}_n\rangle = \sqrt{2E_{\vec{p}_1} \cdot 2E_{\vec{p}_2} \cdot ... \cdot 2E_{\vec{p}_n}} \hat{a}^\dagger_{\vec{p}_1} \hat{a}^\dagger_{\vec{p}_2} ... \hat{a}^\dagger_{\vec{p}_n} |0\rangle .$$

Since a creation operator commutes with any other creation operator, it does not matter in which order we write the creation operators. In order to have a manifest symmetric expressions we can equally well write

$$|\vec{p}_1, \vec{p}_2, ..., \vec{p}_n\rangle = \sqrt{2E_{\vec{p}_1} \cdot 2E_{\vec{p}_2} \cdot ... \cdot 2E_{\vec{p}_n}} \frac{1}{n!} \sum_{\sigma \in S_n} \hat{a}^\dagger_{\vec{p}_{\sigma_1}} \hat{a}^\dagger_{\vec{p}_{\sigma_2}} ... \hat{a}^\dagger_{\vec{p}_{\sigma_n}} |0\rangle ,$$

where the sum is over $n!$ permutations of the set $\{1, 2, ..., n\}$.

5.3 The Fock space

We have already mentioned, that all states of the system can be reached by succesively applying creation operators $\hat{a}^\dagger_{\vec{p}}$ to the ground state $|0\rangle$. We can formalise this situation. From quantum mechanics we are familiar with the concept of a Hilbert space. We recall that a Hilbert space is a vectorspace, which has an inner product and which is complete. As base field we will always taks the complex numbers $\mathbb{C}$. Let us denote by V_1 the Hilbert space of the one-particle states

$$V_1 = \left\{ |\vec{p}\rangle \mid \vec{p} \in \mathbb{R}^3 \right\}.$$

For convenience we denote

$$V_0 = \mathbb{C}.$$

We further set

$$V_n = \text{Sym } (V_1)^{\otimes n},$$

i.e. V_n is the symmetrised n-fold tensor product of V_1. Symmetrisation means for example that $|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle$ and $|\vec{p}_2\rangle \otimes |\vec{p}_1\rangle$ are identified in V_2 and we simply write

$$|\vec{p}_1, \vec{p}_2\rangle = |\vec{p}_1\rangle \otimes_{\text{sym}} |\vec{p}_2\rangle = |\vec{p}_2\rangle \otimes_{\text{sym}} |\vec{p}_1\rangle \in V_2.$$

The **Fock space** of the system is the direct sum of all V_n, where $n \in \mathbb{N}_0$.

$$V = \bigoplus_{n=0}^{\infty} V_n = V_0 \oplus V_1 \oplus V_2 \oplus ...$$

The Fock space is a Hilbert space. V_0 contains the zero-particle states, V_1 the one-particles state, V_2 the two-particle states, etc.. Thus the states of our system form a Fock space.

5.4 The Heisenberg picture

We now turn to the Heisenberg picture. In this section we will denote for clarity operators in the Heisenberg picture by $\hat{O}_H(x)$ and operators in the Schrödinger picture by $\hat{O}_S(\vec{x})$. In later section we will drop the subscripts H and S and denote the operators in the Heisenberg picture simply by $\hat{O}(x)$ and the operators in the Schrödinger picture by $\hat{O}(\vec{x})$. Note that the operators in the Heisenberg picture depend on the four-vector x, while the operators in the Schrödinger picture depend only on the spatial vector $\vec{x}$, therefore it is clear what operator is meant.

In the Heisenberg picture, the operators $\hat{\phi}_H(x)$ and $\hat{\pi}_H(x)$ are now time-dependent:

$$\begin{aligned} \hat{\phi}_H(x) &= e^{i\hat{H}t}\hat{\phi}_S(\vec{x})e^{-i\hat{H}t}, \\ \hat{\pi}_H(x) &= e^{i\hat{H}t}\hat{\pi}_S(\vec{x})e^{-i\hat{H}t}. \end{aligned}$$

If the Hamilton operator is time-independent we have $\hat{H}_H = \hat{H}_S = \hat{H}$. Furthermore we have

$$T \exp\left(-i\int\limits_0^t dt'\hat{H}\right) = e^{-i\hat{H}t}.$$

The states in the Heisenberg picture are given by

$$|X\rangle_H = e^{i\hat{H}t}\,|X,t\rangle_S.$$

At $t = 0$ the states in the Heisenberg picture and the Schrödinger picture agree: $|X\rangle_H = |X,0\rangle_S$. Sometimes it is useful to generalise this slightly , such that the states in the Heisenberg picture and the Schrödinger picture agree at t_0. The transformation formulae in this case read

$$\hat{O}_H(x) = e^{i\hat{H}(t-t_0)}\hat{O}_S(\vec{x})e^{-i\hat{H}(t-t_0)}, \qquad |X\rangle_H = e^{i\hat{H}(t-t_0)}\,|X,t\rangle_S.$$

Note that the Heisenberg operator $\hat{O}_H$ depends in this case on t and t_0 (to be concrete: $\hat{O}_H$ depends on the difference $t-t_0$) and that the Heisenberg state $|X\rangle_H$ is always independent of t:

$$i\frac{\partial}{\partial t}|X\rangle_H = 0.$$

From the Heisenberg equation of motion

$$i\frac{\partial}{\partial t}\hat{O}_H = \left[\hat{O}_H, \hat{H}\right],$$

we find

$$\begin{aligned} i\frac{\partial}{\partial t}\hat{\phi}_H(x) &= i\hat{\pi}_H(x), \\ i\frac{\partial}{\partial t}\hat{\pi}_H(x) &= i\left(\vec{\nabla}^2 - m^2\right)\hat{\phi}_H(x). \end{aligned}$$

Combining these two results yields

$$\frac{\partial^2}{\partial t^2}\hat{\phi}_H(x) \;=\; \left(\vec{\nabla}^2 - m^2\right)\hat{\phi}_H(x),$$

or

$$\left(\Box + m^2\right)\hat{\phi}_H(x) \;=\; 0.$$

Thus, the operator $\hat{\phi}_H(x)$ fulfills the Klein-Gordon equation.

For a better understanding we express $\hat{\phi}_H(x)$ and $\hat{\pi}_H(x)$ in terms of creation and annihilation operators. From

$$\left[\hat{H}, \hat{a}^\dagger_{\vec{p}}\right] \;=\; E_{\vec{p}}\,\hat{a}^\dagger_{\vec{p}}, \qquad \left[\hat{H}, \hat{a}_{\vec{p}}\right] \;=\; -E_{\vec{p}}\,\hat{a}_{\vec{p}},$$

we have

$$\hat{H}\hat{a}^\dagger_{\vec{p}} \;=\; \hat{a}^\dagger_{\vec{p}}\left(\hat{H} + E_{\vec{p}}\right), \qquad \hat{H}\hat{a}_{\vec{p}} \;=\; \hat{a}_{\vec{p}}\left(\hat{H} - E_{\vec{p}}\right),$$

and therefore

$$\hat{H}^n\hat{a}^\dagger_{\vec{p}} \;=\; \hat{a}^\dagger_{\vec{p}}\left(\hat{H} + E_{\vec{p}}\right)^n, \qquad \hat{H}^n\hat{a}_{\vec{p}} \;=\; \hat{a}_{\vec{p}}\left(\hat{H} - E_{\vec{p}}\right)^n.$$

Therefore

$$e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}t} \;=\; \hat{a}^\dagger_{\vec{p}}e^{iE_{\vec{p}}t}, \qquad e^{i\hat{H}t}\hat{a}_{\vec{p}}e^{-i\hat{H}t} \;=\; \hat{a}_{\vec{p}}e^{-iE_{\vec{p}}t}.$$

From

$$\begin{aligned}
\hat{\phi}_S(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\hat{a}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right), \\
\hat{\pi}_S(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_{\vec{p}}}{2}}\left(\hat{a}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - \hat{a}^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right)
\end{aligned}$$

we then obtain

$$\begin{aligned}
\hat{\phi}_H(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left.\left(\hat{a}_{\vec{p}}e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}}e^{ip\cdot x}\right)\right|_{p^0=E_{\vec{p}}}, \\
\hat{\pi}_H(x) &= \frac{\partial}{\partial t}\hat{\phi}_H(x).
\end{aligned}$$

Remark: $\hat{a}_{\vec{p}}$ and $\hat{a}^\dagger_{\vec{p}}$ denote always the time-independent Schrödinger-picture ladder operators. The time-dependent ladder operators in the Heisenberg picture are

$$\hat{a}^\dagger_{\vec{p},H}(t) \;=\; e^{i\hat{H}t}\hat{a}^\dagger_{\vec{p}}e^{-i\hat{H}t} \;=\; \hat{a}^\dagger_{\vec{p}}e^{iE_{\vec{p}}t}, \qquad \hat{a}_{\vec{p},H}(t) \;=\; e^{i\hat{H}t}\hat{a}_{\vec{p}}e^{-i\hat{H}t} \;=\; \hat{a}_{\vec{p}}e^{-iE_{\vec{p}}t}.$$

The subscript H indicates Heisenberg operators.

Remark 2: The above equation makes the duality between particle and wave interpretations of the quantum field explicit: On the one hand, $\hat{\phi}_H(x)$ is written as a Hilbert space operator, which creates and destroys the particles that are the quanta of the field excitations. On the other hand, $\hat{\phi}_H(x)$ is written as a linear combination of plane-wave solutions of the Klein-Gordon equation.

5.4.1 Causality

From this section onwards we drop the subscript H for the operators in the Heisenberg picture.

Let us consider in the Heisenberg picture the amplitude for a particle to propagate from y to x:

$$D(x-y) \quad = \quad \left\langle 0 \left| \hat{\phi}(x)\hat{\phi}(y) \right| 0 \right\rangle.$$

We express $\hat{\phi}(x)$ and $\hat{\phi}(y)$ in terms of creation and annihilation operators. Since $\hat{a}_{\vec{q}}$ annihilates the ground state $|0\rangle$ and $\hat{a}^{\dagger}_{\vec{p}}$ when acting to the left annihilates the bra-vector $\langle 0|$, only the term

$$\left\langle 0 \left| \hat{a}_{\vec{p}} \hat{a}^{\dagger}_{\vec{q}} \right| 0 \right\rangle \quad = \quad (2\pi)^3 \delta^3(\vec{p}-\vec{q})$$

survives and we obtain

$$\left\langle 0 \left| \hat{\phi}(x)\hat{\phi}(y) \right| 0 \right\rangle \quad = \quad \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

Remark: This expression is Lorentz-invariant.

Let us first consider the case where the difference $x-y$ is purely in the time-like direction: $x^0 - y^0 = t, \vec{x}-\vec{y} = 0$:

$$\begin{aligned} D(x-y) \quad &= \quad \frac{4\pi}{(2\pi)^3} \int\limits_0^\infty d|\vec{p}| \frac{|\vec{p}|^2}{2\sqrt{|\vec{p}|^2+m^2}} e^{-it\sqrt{|\vec{p}|^2+m^2}} = \frac{1}{4\pi^2} \int\limits_m^\infty dE \sqrt{E^2-m^2} e^{-iEt} \\ &\overset{t\to\infty}{\sim} \quad e^{-imt}. \end{aligned}$$

Let us now consider the case where $x-y$ is purely spatial: $x^0-y^0 = 0, \vec{x}-\vec{y} = \vec{r}$:

$$\begin{aligned} D(x-y) \quad &= \quad \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}} = \frac{2\pi}{(2\pi)^3} \int\limits_0^\infty d|\vec{p}| \int\limits_{-1}^{1} d\cos\theta \frac{|\vec{p}|^2}{2E_{\vec{p}}} e^{i|\vec{p}|r\cos\theta} \\ &= \quad \frac{1}{(2\pi)^2} \int\limits_0^\infty d|\vec{p}| \frac{|\vec{p}|^2}{2E_{\vec{p}}} \frac{e^{i|\vec{p}|r} - e^{-i|\vec{p}|r}}{i|\vec{p}|r} = \frac{-i}{2(2\pi)^2 r} \int\limits_{-\infty}^{\infty} d|\vec{p}| \frac{|\vec{p}| e^{i|\vec{p}|r}}{\sqrt{|\vec{p}|^2+m^2}} \end{aligned}$$

This integral has branch cuts on the imaginary axis starting at $\pm im$. We can deform the contour to go around the upper branch cut, the quarter circles at infinity will give a vanishing contribution. With the substitution $\rho = -i|\vec{p}|$ we obtain

$$\frac{1}{4\pi^2 r} \int\limits_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \quad \overset{r\to\infty}{\sim} \quad e^{-mr}.$$

We find that the propagation amplitude for space-like distances is exponentially vanishing, but non-zero. Is this a problem with causality ? No, to discuss causality we should not ask whether particles can propagate over space-like distances, but whether a measurement performed at one point can affect a measurement at another point whose separation from the first is space-like. The simplest thing to measure is the field $\hat{\phi}(x)$, so let's have a look at the commutator $[\hat{\phi}(x), \hat{\phi}(y)]$, if this commutator vanishes for space-like distances, one measurement cannot affect another one separated at a space-like distance.

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\phi}(y)\right] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \left[\left(\hat{a}_{\vec{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}} e^{ip\cdot x} \right), \left(\hat{a}_{\vec{q}} e^{-iq\cdot y} + \hat{a}^\dagger_{\vec{q}} e^{iq\cdot y} \right) \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\
&= D(x-y) - D(y-x).
\end{aligned}$$

When $(x-y)^2 < 0$, we can perform a Lorentz-transformation on the second term (since each term is separately Lorentz-invariant), taking

$$(x-y) \rightarrow -(x-y).$$

The two terms are therefore equal and cancel in the sum. Therefore causality is preserved.

Remark: If $(x-y)^2 > 0$ there is no continuous Lorentz-transformation, which takes $(x-y) \rightarrow -(x-y)$.

5.4.2 The Klein-Gordon propagator

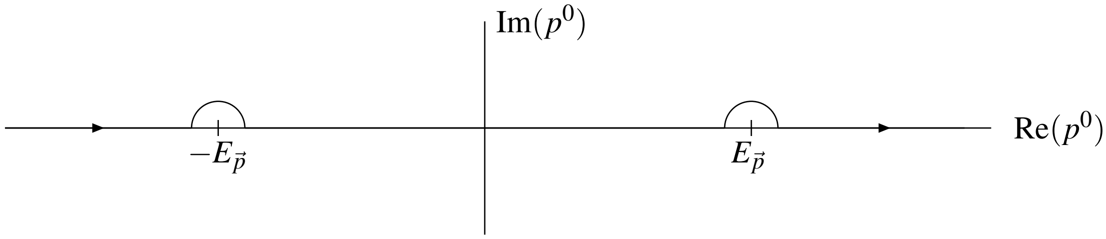
Let us study the commutator $[\hat{\phi}(x), \hat{\phi}(y)]$ a little bit further. Since it is a c-number, we have

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \left\langle 0 \middle| \left[\hat{\phi}(x), \hat{\phi}(y)\right] \middle| 0 \right\rangle.$$

Let us assume that $x^0 > y^0$. Then we have

$$\begin{aligned}
\left\langle 0 \middle| \left[\hat{\phi}(x), \hat{\phi}(y)\right] \middle| 0 \right\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0=-E_{\vec{p}}} \right\} \\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)}.
\end{aligned}$$

In the second term of the second line we changed the integration variables as $\vec{p} \rightarrow -\vec{p}$. In going from the second to the third line we applied Cauchy's theorem in the reverse direction: We first recognise the sum of the two terms in the second line as a sum of residues. Then we re-write the sum of residues as a contour integral. The p^0 integral is to be performed along the contour



The condition $x^0 > y^0$ ensures that the half-circle at infinity in the lower complex plane gives a vanishing contribution. To keep track of the contour we also write

$$\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i}{(p^0 + i\varepsilon)^2 - |\vec{p}|^2 - m^2} e^{-ip\cdot(x-y)},$$

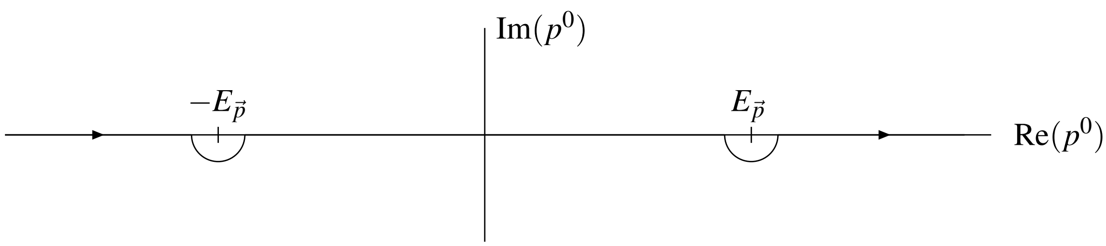
where ε is an infinitessimal quantity with $\varepsilon > 0$. We define

$$D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0 + i\varepsilon)^2 - |\vec{p}|^2 - m^2} e^{-ip\cdot(x-y)}.$$

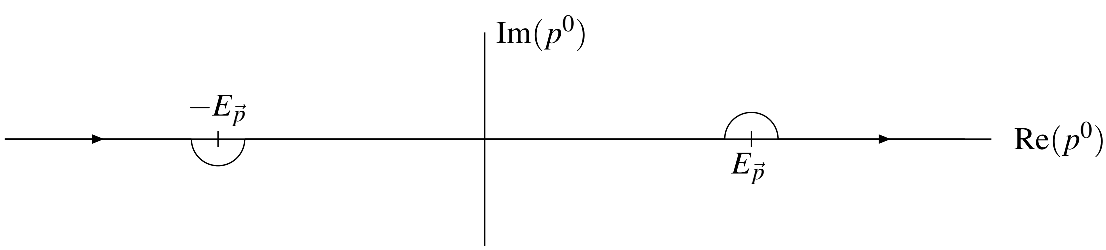
$D_R(x-y)$ is called the **retarded Green's function**. Similar the **advanced Green's function** is defined by

$$D_A(x-y) = -\theta(y^0 - x^0) \left\langle 0 \left| \left[\phi(\hat{x}), \phi(\hat{y}) \right] \right| 0 \right\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0 - i\varepsilon)^2 - |\vec{p}|^2 - m^2} e^{-ip\cdot(x-y)},$$

where in the last expression the contour integral is now evaluated with the contour



There are four possible contours to evaluate the p^0-integration. A third one is of particular importance:



This corresponds to the integral

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)},$$

Again, the $+i\varepsilon$-prescription ensures that the poles are avoided as shown in the figure above.

If $x^0 > y^0$ we can close the contour below and obtain $D(x-y)$. If $x^0 < y^0$ we close the contour above and obtain $D(y-x)$. Therefore

$$\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & \text{for} \quad x^0 > y^0 \\ D(y-x) & \text{for} \quad x^0 < y^0 \end{cases} \\ &= \theta(x^0 - y^0) \left\langle 0 \left| \hat{\phi}(x)\hat{\phi}(y) \right| 0 \right\rangle + \theta(y^0 - x^0) \left\langle 0 \left| \hat{\phi}(y)\hat{\phi}(x) \right| 0 \right\rangle \\ &= \left\langle 0 \left| T\hat{\phi}(x)\hat{\phi}(y) \right| 0 \right\rangle \end{aligned}$$

In the last expression the **time-ordering symbol** T occurs, which orders operators from right to left in non-decreasing time. Thus

$$T\hat{\phi}(x)\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & \text{for} \quad x^0 > y^0, \\ \hat{\phi}(y)\hat{\phi}(x) & \text{for} \quad x^0 < y^0. \end{cases}$$

The quantity $D_F(x-y)$ is called the **Feynman propagator**. Let's have a look at

$$\begin{aligned} \left(\Box + m^2\right) D_F(x-y) &= \left(\Box + m^2\right) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} (-p^2 + m^2) e^{-ip\cdot(x-y)} \\ &= -i\delta^4(x-y). \end{aligned}$$

Therefore, if we look at the Fourier transform $\tilde{D}_F(p)$

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{D}_F(p),$$

we obtain an algebraic equation for $\tilde{D}_F(p)$:

$$(p^2 - m^2)\tilde{D}_F(p) = i.$$

Summary: The Feynman propagator in momentum space is obtained from an algebraic equation. The integration contour in the p^0-plane is given by the $i\varepsilon$-prescription. For the Klein-Gordon propagator we have

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}.$$

5.5 Wick's theorem

We already had the definition of the time-ordered product: This product orders operators such that the time does not decrease from right to left.

$$T\hat{\phi}(x)\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & \text{for } x^0 > y^0, \\ \hat{\phi}(y)\hat{\phi}(x) & \text{for } y^0 > x^0. \end{cases}$$

In addition we introduce the **normal product**, which orders operators such that all annihilation operators are on the right of all creation operators:

$$\begin{aligned} :\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{q}}: &= \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{q}}, \\ :\hat{a}_{\vec{q}}\hat{a}^\dagger_{\vec{p}}: &= \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{q}}. \end{aligned}$$

We notice that the vacuum expectation value of a normal ordered product is zero (unless the product is empty):

$$\left\langle 0 \right| :\hat{\phi}(x_1)\hat{\phi}(x_2)....\hat{\phi}(x_n): \left| 0 \right\rangle = 0.$$

We further introduce the so-called "contraction", which is just the vacuum expectation value of the time-ordered product of two operators (or equivalently the Feynman propagator $D_F(x-y)$).

$$\underbrace{\hat{\phi}(x)\hat{\phi}(y)} = \left\langle 0 \right| T\hat{\phi}(x)\hat{\phi}(y) \left| 0 \right\rangle = D_F(x-y).$$

Wick's theorem states that

$$T\hat{\phi}(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n) = :\hat{\phi}(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n): + \text{ all possible contractions.}$$

Example:

$$\begin{aligned} T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) =& :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4): \\ &+ :\underbrace{\hat{\phi}(x_1)\hat{\phi}(x_2)}\hat{\phi}(x_3)\hat{\phi}(x_4): + :\underbrace{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)}\hat{\phi}(x_4): + :\underbrace{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)}: \\ &+ :\hat{\phi}(x_1)\underbrace{\hat{\phi}(x_2)\hat{\phi}(x_3)}\hat{\phi}(x_4): + :\hat{\phi}(x_1)\underbrace{\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)}: + :\hat{\phi}(x_1)\hat{\phi}(x_2)\underbrace{\hat{\phi}(x_3)\hat{\phi}(x_4)}: \\ &+ :\underbrace{\hat{\phi}(x_1)\hat{\phi}(x_2)}\underbrace{\hat{\phi}(x_3)\hat{\phi}(x_4)}: + :\underbrace{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)}_{}\hspace{-2em}\underbrace{\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)}: + :\underbrace{\hat{\phi}(x_1)\underbrace{\hat{\phi}(x_2)\hat{\phi}(x_3)}\hat{\phi}(x_4)}: \end{aligned}$$

Proof: We decompose any operator into positive and negative frequency parts:

$$\begin{aligned} \hat{\phi}(x) &= \hat{\phi}^+(x) + \hat{\phi}^-(x), \\ \hat{\phi}^+(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \hat{a}_{\vec{p}} e^{-ip\cdot x}, \quad \hat{\phi}^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \hat{a}^\dagger_{\vec{p}} e^{ip\cdot x}. \end{aligned}$$

$\hat{\phi}^+$ contains only annihilation operators, $\hat{\phi}^-$ contains only creation operators. We proof Wick's theorem by induction. We start at $n = 2$ and assume $x^0 > y^0$:

$$
\begin{aligned}
T\hat{\phi}(x)\hat{\phi}(y) &= \hat{\phi}(x)\hat{\phi}(y) \\
&= \hat{\phi}^+(x)\hat{\phi}^+(y) + \hat{\phi}^+(x)\hat{\phi}^-(y) + \hat{\phi}^-(x)\hat{\phi}^+(y) + \hat{\phi}^-(x)\hat{\phi}^-(y) \\
&= \hat{\phi}^+(x)\hat{\phi}^+(y) + \hat{\phi}^-(y)\hat{\phi}^+(x) + \hat{\phi}^-(x)\hat{\phi}^+(y) + \hat{\phi}^-(x)\hat{\phi}^-(y) + \left[\hat{\phi}^+(x), \hat{\phi}^-(y)\right] \\
&= :\hat{\phi}(x)\hat{\phi}(y): + \left[\hat{\phi}^+(x), \hat{\phi}^-(y)\right] \\
&= :\hat{\phi}(x)\hat{\phi}(y): + \left\langle 0 \middle| \left[\hat{\phi}^+(x), \hat{\phi}^-(y)\right] \middle| 0 \right\rangle \\
&= :\hat{\phi}(x)\hat{\phi}(y): + D_F(x-y) \\
&= :\hat{\phi}(x)\hat{\phi}(y): + \underbrace{\hat{\phi}(x)\hat{\phi}}(y) .
\end{aligned}
$$

The proof for the case $n = 2$ and $x^0 < y^0$ is similar. Not let's assume that it is valid for $n-1$ fields. Again we assume $x_1^0 > x_2^0 > ... > x_n^0$.

$$
\begin{aligned}
&T\hat{\phi}(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n) = \phi(x_1)\phi(x_2)...\phi(x_n) \\
&\quad = \hat{\phi}(x_1)\, T\hat{\phi}(x_2)...\hat{\phi}(x_n) \\
&\quad = \hat{\phi}(x_1) : \left\{\hat{\phi}(x_2)...\hat{\phi}(x_n) + \text{all contractions not involving } \hat{\phi}(x_1)\right\} : \\
&\quad = \left(\hat{\phi}^+(x_1) + \hat{\phi}^-(x_1)\right) : \left\{\hat{\phi}(x_2)...\hat{\phi}(x_n) + \text{all contractions not involving } \hat{\phi}(x_1)\right\} :
\end{aligned}
$$

We want to move $\hat{\phi}^+(x_1)$ and $\hat{\phi}^-(x_1)$ inside the normal product. For $\hat{\phi}^-(x_1)$ this is easy: $\hat{\phi}^-(x_1)$ contains only creation operators, therefore

$$
\hat{\phi}^-(x_1) : \hat{\phi}(x_2)...\hat{\phi}(x_n) : \quad = \quad : \hat{\phi}^-(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n) :,
$$

and we can just move it in to the left of all other operators. On the other hand we have

$$
: \hat{\phi}^+(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n) : \quad = \quad : \hat{\phi}(x_2)...\hat{\phi}(x_n)\hat{\phi}^+(x_1) :,
$$

therefore

$$
\begin{aligned}
&\hat{\phi}^+(x_1) : \hat{\phi}(x_2)...\hat{\phi}(x_n) : \;=\; : \hat{\phi}(x_2)\hat{\phi}(x_3)...\hat{\phi}(x_n) : \hat{\phi}^+(x_1) \;+\; \left[\hat{\phi}^+(x_1), : \hat{\phi}(x_2)\hat{\phi}(x_3)...\hat{\phi}(x_n) :\right] \\
&\quad = \; : \hat{\phi}^+(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)...\hat{\phi}(x_n) : \;+\; : \left[\hat{\phi}^+(x_1), \hat{\phi}(x_2)\right] \hat{\phi}(x_3)...\hat{\phi}(x_n) : \\
&\qquad + \; : \hat{\phi}(x_2) \left[\hat{\phi}^+(x_1), \hat{\phi}(x_3)\right] ...\hat{\phi}(x_n) : \;+\; : \hat{\phi}(x_2)\hat{\phi}(x_3)... \left[\hat{\phi}^+(x_1), \hat{\phi}(x_n)\right] : \\
&\quad = \; : \hat{\phi}^+(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)...\hat{\phi}(x_n) : \;+\; : \underbrace{\hat{\phi}^+(x_1)\hat{\phi}}(x_2)\hat{\phi}(x_3)...\hat{\phi}(x_n) : + : \underbrace{\hat{\phi}^+(x_1)\hat{\phi}(x_2)\hat{\phi}}(x_3)...\hat{\phi}(x_n) : \\
&\qquad + \; ... \; + \; : \underbrace{\hat{\phi}^+(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)...\hat{\phi}}(x_n) :,
\end{aligned}
$$

which proves Wick's theorem. Let us now apply Wick's theorem to the vacuum expectation value of

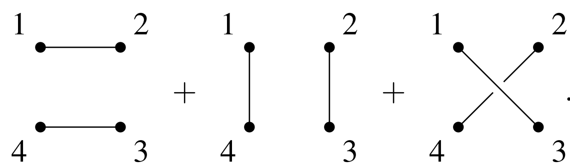
$$
\left\langle 0 \middle| T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) \middle| 0 \right\rangle .
$$

By construction, the vacuum expectation value of a non-empty normal product vanishes, therefore

$$
\begin{aligned}
&\langle 0 | T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) | 0 \rangle = \\
&= \left\langle 0 \left| :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4): \right| 0 \right\rangle + \left\langle 0 \left| :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4): \right| 0 \right\rangle \\
&\quad + \left\langle 0 \left| :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4): \right| 0 \right\rangle \\
&= \langle 0 | T\hat{\phi}(x_1)\hat{\phi}(x_2) | 0 \rangle \langle 0 | T\hat{\phi}(x_3)\hat{\phi}(x_4) | 0 \rangle + \langle 0 | T\hat{\phi}(x_1)\hat{\phi}(x_3) | 0 \rangle \langle 0 | T\hat{\phi}(x_2)\hat{\phi}(x_4) | 0 \rangle \\
&\quad + \langle 0 | T\hat{\phi}(x_1)\hat{\phi}(x_4) | 0 \rangle \langle 0 | T\hat{\phi}(x_2)\hat{\phi}(x_3) | 0 \rangle \\
&= D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3).
\end{aligned}
$$

Graphically,

$$\langle 0 | T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) | 0 \rangle =$$



5.6 Interacting fields

Up to now we considered "free" fields, e.g. fields without any interactions. For the free Klein-Gordon field we had the Lagrange density

$$\mathcal{L}_0 = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2.$$

In this theory, no interactions and no scattering occurs. Let us start to look at more interesting theories with interactions:

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4,$$

where λ is a dimensionless coupling constant. This theory is often called "phi-fourth" theory and one of the simplest theories with interactions. Obviously,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -\frac{\lambda}{4!} \phi^4.$$

The "classical" equation of motion for the ϕ^4 theory is

$$\left(\Box + m^2\right) \phi = -\frac{\lambda}{3!} \phi^3,$$

which cannot be solved by Fourier analysis as the free Klein-Gordon equation. For $\lambda = 0$ we recover the Klein-Gordon equation. If λ is small we may treat the interacting theory by perturbation theory.

Let us now look at the quantum theory. In this chapter we use for clarity in the essential places the subscripts S, H and I, to denote operators in the Schrödinger picture, the Heisenberg picture and the interaction picture, respectively. As $\mathcal{L}_{\text{int}}$ does not involve any derivatives, the definition of

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

is unaffected by $\mathcal{L}_{\text{int}}$. With the same reasoning, we still impose in the quantum theory equal-time commutation relations

$$\begin{aligned}
\left[\hat{\phi}_S(\vec{x}), \hat{\pi}_S(\vec{y})\right] &= i\delta^3\left(\vec{x}-\vec{y}\right), \\
\left[\hat{\phi}_S(\vec{x}), \hat{\phi}_S(\vec{y})\right] &= \left[\hat{\pi}_S(\vec{x}), \hat{\pi}_S(\vec{y})\right] = 0.
\end{aligned}$$

We can write the full Hamiltonian as

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \hat{H}_{\text{int}}, \\
\hat{H}_0 &= \int d^3x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\nabla\hat{\phi}\right)^2 + \frac{1}{2}m^2\hat{\phi}^2\right], \\
\hat{H}_{\text{int}} &= \int d^3x \frac{\lambda}{4!}\phi^4.
\end{aligned}$$

Recall that as long as $\hat{H}$ does not depend explicitly on the time, we have $\hat{H} = \hat{H}_S = \hat{H}_H$. We start with the study of the two-point correlation function, or the two-point Green's function

$$\left\langle \Omega \middle| T\hat{\phi}_H(x)\hat{\phi}_H(y) \middle| \Omega \right\rangle.$$

$|\Omega\rangle$ denotes the ground state of the interacting theory, which is in generally different from the ground state $|0\rangle$ of the free theory. The interaction Hamiltonian enters in two places: In the definition of $|\Omega\rangle$ and in the definition of the Heisenberg field

$$\hat{\phi}_H(x) = e^{i\hat{H}(t-t_0)}\hat{\phi}_S(\vec{x})e^{-i\hat{H}(t-t_0)}.$$

It is easiest to begin with the Heisenberg field $\hat{\phi}_H(x)$. At any fixed time t_0 we can of course expand the Schrödinger field as before in terms of creation and annihilation operators:

$$\hat{\phi}_S(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right).$$

For $t \neq t_0$ we have in the Heisenberg picture

$$\hat{\phi}_H(t, \vec{x}) = e^{i\hat{H}(t-t_0)}\hat{\phi}_S(t_0, \vec{x})e^{-i\hat{H}(t-t_0)}.$$

For $\lambda = 0$, $\hat{H}$ reduces to $\hat{H}_0$:

$$\hat{\phi}_H(t, \vec{x})\big|_{\lambda=0} = e^{i\hat{H}_0(t-t_0)}\hat{\phi}_S(t_0, \vec{x})e^{-i\hat{H}_0(t-t_0)}.$$

When λ is small, this expression will still give the most important part of the time dependence of $\hat{\phi}_H(x)$, and thus it is convenient to give this quantity a name: the field in the **interaction picture**, $\hat{\phi}_I(x)$.

$$\hat{\phi}_I(x) = e^{i\hat{H}_0(t-t_0)}\hat{\phi}_S(t_0,\vec{x})e^{-i\hat{H}_0(t-t_0)}.$$

The states in the interaction picture are defined by

$$|X,t\rangle_I = e^{i\hat{H}_0(t-t_0)}|X,t\rangle_S.$$

At $t = t_0$, the states in the interaction picture and in the Schrödinger picture agree: $|X,t_0\rangle_I = |X,t_0\rangle_S$. This is in complete analogy with quantum mechanics, where one has apart from the Schrödinger and Heisenberg picture also the interaction picture. As in the free theory we find

$$\hat{\phi}_I(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(\hat{a}_{\vec{p}}e^{-ip\cdot x'} + \hat{a}^\dagger_{\vec{p}}e^{ip\cdot x'}\right)\Big|_{p^0=E_{\vec{p}}=\sqrt{|\vec{p}|^2+m^2},x'^0=t-t_0,\vec{x}'=\vec{x}}.$$

The problem is now to express the full Heisenberg field $\hat{\phi}_H(x)$ in terms of $\hat{\phi}_I(x)$. We have

$$\begin{aligned}\hat{\phi}_H(x) &= e^{i\hat{H}(t-t_0)}e^{-i\hat{H}_0(t-t_0)}\hat{\phi}_I(x)e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}\\ &= \hat{U}^\dagger(t,t_0)\hat{\phi}_I(x)\hat{U}(t,t_0),\end{aligned}$$

where

$$\hat{U}(t,t_0) = e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}.$$

We have

$$\hat{U}(t_0,t_0) = 1$$

and

$$\begin{aligned}i\frac{\partial}{\partial t}\hat{U}(t,t_0) &= e^{i\hat{H}_0(t-t_0)}\left(\hat{H}-\hat{H}_0\right)e^{-i\hat{H}(t-t_0)}\\ &= e^{i\hat{H}_0(t-t_0)}\hat{H}_{\text{int}}e^{-i\hat{H}(t-t_0)}\\ &= \underbrace{e^{i\hat{H}_0(t-t_0)}\hat{H}_{\text{int}}e^{-i\hat{H}_0(t-t_0)}}_{\hat{H}_I(t)}\ \underbrace{e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t_0)}}_{\hat{U}(t,t_0)}.\end{aligned}$$

$\hat{H}_I(t)$ is the interaction Hamiltonian written in the interaction picture:

$$\hat{H}_I(t) = e^{i\hat{H}_0(t-t_0)}\hat{H}_{\text{int}}e^{-i\hat{H}_0(t-t_0)} = \int d^3x\frac{\lambda}{4!}\hat{\phi}_I^4.$$

Therefore

$$i\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0), \quad \hat{U}(t_0,t_0) = 1.$$

A solution is given by

$$\hat{U}(t,t_0) = 1+(-i)\int_{t_0}^{t} dt_1\, \hat{H}_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{H}_I(t_1)\hat{H}_I(t_2)$$
$$+(-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3\, \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3) + \ldots$$

This solution can be verified by differentiation. The initial condition $\hat{U}(t_0,t_0) = 1$ is obviously satisfied. Note that the various factors of $\hat{H}_I$ stand in time order, later on the left. Note that

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{H}_I(t_1)\hat{H}_I(t_2) = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\left(\hat{H}_I(t_1)\hat{H}_I(t_2)\right).$$

Similar

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}_I(t_1)\hat{H}_I(t_2)\ldots\hat{H}_I(t_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n\, T\left(\hat{H}_I(t_1)\hat{H}_I(t_2)\ldots\hat{H}_I(t_n)\right).$$

Therefore

$$\hat{U}(t,t_0) = 1+(-i)\int_{t_0}^{t} dt_1\, \hat{H}_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\left(\hat{H}_I(t_1)\hat{H}_I(t_2)\right)$$
$$+\frac{(-i)^3}{3!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3\, T\left(\hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3)\right) + \ldots$$
$$= T\left\{\exp\left[-i\int_{t_0}^{t} dt'\, \hat{H}_I(t')\right]\right\}.$$

Therefore we have expressed the full field $\hat{\phi}_H$ in terms of $\hat{\phi}_I$;

$$\hat{\phi}_H(x) = \hat{U}^{\dagger}(t,t_0)\hat{\phi}_I(x)\hat{U}(t,t_0),$$

We can generalise the evolution operator to take as the second argument values other than our reference time t_0:

$$\hat{U}(t,t') = T\left\{\exp\left[-i\int_{t'}^{t} dt''\, \hat{H}_I(t'')\right]\right\}.$$

$\hat{U}(t,t')$ satisfies

$$i\frac{\partial}{\partial t}\hat{U}(t,t') = \hat{H}_I(t)\hat{U}(t,t'), \qquad \hat{U}(t,t) = 1,$$

and the identities

$$\begin{aligned}\hat{U}(t_1,t_2)\hat{U}(t_2,t_3) &= \hat{U}(t_1,t_3),\\ \hat{U}(t_1,t_3)\hat{U}^\dagger(t_2,t_3) &= \hat{U}(t_1,t_2).\end{aligned}$$

$\hat{U}(t,t')$ is unitary:

$$\hat{U}(t,t')\hat{U}^\dagger(t,t') = \hat{U}(t,t) = 1.$$

We further have

$$\begin{aligned}\hat{U}(t,t') &= \hat{U}(t,t_0)\hat{U}^\dagger(t',t_0)\\ &= e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t')}e^{-i\hat{H}_0(t'-t_0)},\end{aligned}$$

In particular we have

$$\hat{U}(t_0,-T) = e^{-i\hat{H}(t_0-(-T))}e^{-i\hat{H}_0((-T)-t_0)}.$$

Let us now discuss the ground state $|\Omega\rangle$. Imagine starting with $|0\rangle$, the ground state of $\hat{H}_0$ and evolving through time with $\hat{H}$:

$$e^{-i\hat{H}T}|0\rangle = \sum_n e^{-iE_n^{\text{full}}T}|n^{\text{full}}\rangle\langle n^{\text{full}}|0\rangle.$$

Here we inserted a complete set of states. E_n^{full} and $|n^{\text{full}}\rangle$ are the eigenvalues and eigenstates of the full Hamiltonian $\hat{H}$. The eigenvalues and eigenstates of the free Hamiltonian will be denoted by E_n and $|n\rangle$. We define the zero of the energy by

$$\hat{H}_0|0\rangle = 0.$$

We single out the ground state $|\Omega\rangle = |0^{\text{full}}\rangle$:

$$e^{-i\hat{H}T}|0\rangle = e^{-iE_0^{\text{full}}T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n^{\text{full}}T}|n^{\text{full}}\rangle\langle n^{\text{full}}|0\rangle,$$

where $E_0^{\text{full}} = \langle\Omega|\hat{H}|\Omega\rangle$. Since $E_n^{\text{full}} > E_0^{\text{full}}$ for $n > 0$ the additional terms disappear, if we send $T \to \infty(1-i\varepsilon)$. Therefore

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)}\left(e^{-iE_0^{\text{full}}T}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}T}|0\rangle.$$

Since T is now very large, we can shift it by a small constant:

$$\begin{aligned}|\Omega\rangle &= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0^{\text{full}}(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}(T+t_0)}|0\rangle \\ &= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0^{\text{full}}(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-i\hat{H}(t_0-(-T))}e^{-i\hat{H}_0((-T)-t_0)}|0\rangle \\ &= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0^{\text{full}}(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} \hat{U}(t_0,-T)|0\rangle\end{aligned}$$

In the second line we have used $\hat{H}_0|0\rangle = 0$. Similar, we can express $\langle\Omega|$ as

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|\hat{U}(T,t_0)\left(e^{-iE_0^{\text{full}}(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}.$$

Therefore we have for $x^0 > y^0 > t_0$ (we drop the subscript H for $\hat{\phi}(x) = \hat{\phi}_H(x)$):

$$\begin{aligned}&\left\langle\Omega\left|T\hat{\phi}(x)\hat{\phi}(y)\right|\Omega\right\rangle = \\ &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\hat{U}(T,t_0)\hat{U}^\dagger(x^0,t_0)\hat{\phi}_I(x)\hat{U}(x^0,t_0)\hat{U}^\dagger(y^0,t_0)\hat{\phi}_I(x)\hat{U}(y^0,t_0)\hat{U}(t_0,-T)|0\rangle}{e^{-2iE_0^{\text{full}}T}\langle\Omega|0\rangle\langle 0|\Omega\rangle} \\ &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\hat{U}(T,x^0)\hat{\phi}_I(x)\hat{U}(x^0,y^0)\hat{\phi}_I(x)\hat{U}(y^0,-T)|0\rangle}{e^{-2iE_0^{\text{full}}T}\,|\langle 0|\Omega\rangle|^2}\end{aligned}$$

Now

$$1 = \langle\Omega|\Omega\rangle = \frac{\langle 0|\hat{U}(T,t_0)\hat{U}(t_0,-T)|0\rangle}{e^{-2iE_0^{\text{full}}T}\,|\langle 0|\Omega\rangle|^2} = \frac{\langle 0|\hat{U}(T,-T)|0\rangle}{e^{-2iE_0^{\text{full}}T}\,|\langle 0|\Omega\rangle|^2}.$$

Therefore we have for $x^0 > y^0$

$$\left\langle\Omega\left|T\hat{\phi}(x)\hat{\phi}(y)\right|\Omega\right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\hat{U}(T,x^0)\hat{\phi}_I(x)\hat{U}(x^0,y^0)\hat{\phi}_I(x)\hat{U}(y^0,-T)|0\rangle}{\langle 0|\hat{U}(T,-T)|0\rangle}.$$

We notice that all fields are in time-order. The final formula, valid for any x^0 and y^0 reads

$$\begin{aligned}\left\langle\Omega\left|T\hat{\phi}(x)\hat{\phi}(y)\right|\Omega\right\rangle &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle 0\left|T\left\{\hat{\phi}_I(x)\hat{\phi}_I(y)\hat{U}(T,-T)\right\}\right|0\right\rangle}{\left\langle 0\left|\hat{U}(T,-T)\right|0\right\rangle} \\ &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle 0\left|T\left\{\hat{\phi}_I(x)\hat{\phi}_I(y)\exp\left[-i\int\limits_{-T}^{T} dt\,\hat{H}_I(t)\right]\right\}\right|0\right\rangle}{\left\langle 0\left|T\left\{\exp\left[-i\int\limits_{-T}^{T} dt\,\hat{H}_I(t)\right]\right\}\right|0\right\rangle}.\end{aligned}$$

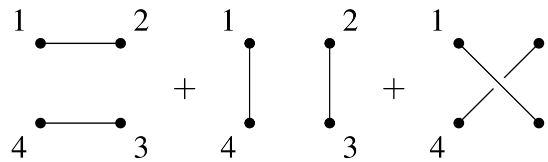
There is an obvious generalisation to more than two fields:

$$\begin{aligned}&\left\langle\Omega\left|T\hat{\phi}(x_1)\hat{\phi}(x_2)...\hat{\phi}(x_n)\right|\Omega\right\rangle = \\ &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle 0\left|T\left\{\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)...\hat{\phi}_I(x_n)\exp\left[-i\int\limits_{-T}^{T} dt\,\hat{H}_I(t)\right]\right\}\right|0\right\rangle}{\left\langle 0\left|T\left\{\exp\left[-i\int\limits_{-T}^{T} dt\,\hat{H}_I(t)\right]\right\}\right|0\right\rangle}.\end{aligned}$$

5.7 Feynman diagrams

We have already seen in the free field theory, that we can represent certain expressions graphically, e.g.

$$\left\langle 0 \middle| T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) \middle| 0 \right\rangle \;=\;$$



This describes two non-interacting particles. Things get more interesting in the interacting theory as soons as we have more than one field at the same space-time point. Let's again look at

$$\left\langle \Omega \middle| T\hat{\phi}(x)\hat{\phi}(y) \middle| \Omega \right\rangle \;=\; \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y) \exp\left[-i \int\limits_{-T}^{T} dt\, \hat{H}_I(t) \right] \right\} \middle| 0 \right\rangle}{\left\langle 0 \middle| T\left\{ \exp\left[-i \int\limits_{-T}^{T} dt\, \hat{H}_I(t) \right] \right\} \middle| 0 \right\rangle}$$

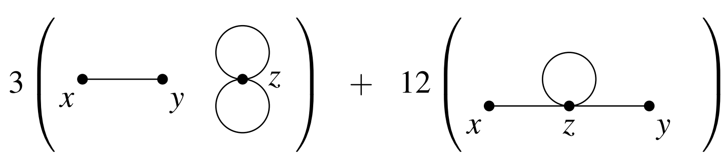
in the interacting ϕ^4-theory. Expanding the exponential in the numerator we obtain

$$\begin{aligned}
&\left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y) \exp\left[-i \int\limits_{-T}^{T} dt\, \hat{H}_I(t) \right] \right\} \middle| 0 \right\rangle = \\
&\qquad \left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y) \right\} \middle| 0 \right\rangle + \left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y) \left[-i \int\limits_{-T}^{T} dt\, \hat{H}_I(t) \right] \right\} \middle| 0 \right\rangle + \dots \\
&= D_F(x-y) + \left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y)(-i) \int dt \int d^3z \frac{\lambda}{4!}\hat{\phi}_I^4 \right\} \middle| 0 \right\rangle + \dots \\
&= D_F(x-y) + \left(\frac{-i\lambda}{4!} \right) \int d^4z \left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y)\hat{\phi}_I(z)\hat{\phi}_I(z)\hat{\phi}_I(z)\hat{\phi}_I(z) \right\} \middle| 0 \right\rangle + \dots
\end{aligned}$$

For the term proportional to λ we use Wick' theorem. In total there are 15 possibilities of contracting the 6 fields with each other. However, only two of them are really different. The first possibility is to contract $\hat{\phi}_I(x)$ with $\hat{\phi}_I(y)$, then we have three possibilities of contracting the four $\hat{\phi}_I(z)$ with each other, all of them are equivalent. The second possibility consists in contracting $\hat{\phi}_I(x)$ with one of the $\hat{\phi}_I(z)$ (four choices) and $\hat{\phi}_I(y)$ with one of the remaining $\hat{\phi}_I(z)$ (three choices). The two remaining $\hat{\phi}_I(z)$ are then contracted with each other. Here we have in total $4 \cdot 3 = 12$ possibilities to produce this configuration. Therefore

$$\begin{aligned}
&\left\langle 0 \middle| T\left\{ \hat{\phi}_I(x)\hat{\phi}_I(y)\hat{\phi}_I(z)\hat{\phi}_I(z)\hat{\phi}_I(z)\hat{\phi}_I(z) \right\} \middle| 0 \right\rangle = \\
&\qquad 3D_F(x-y)D_F(z-z)D_F(z-z) + 12D_F(x-z)D_F(y-z)D_F(z-z).
\end{aligned}$$

Graphically:



Lines in this diagram are called **propagators**, internal points, where four lines meet, are called **vertices**. Therefore

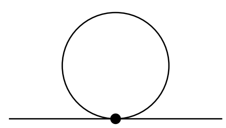
$$\left(\frac{-i\lambda}{4!}\right)\int d^4z\left\langle 0\left|T\left\{\hat{\phi}_I(x)\hat{\phi}_I(y)\hat{\phi}_I(z)\hat{\phi}_I(z)\hat{\phi}_I(z)\hat{\phi}_I(z)\right\}\right|0\right\rangle =$$

$$(-i\lambda)\int d^4z\left[\frac{1}{8}D_F(x-y)D_F(z-z)D_F(z-z)+\frac{1}{2}D_F(x-z)D_F(y-z)D_F(z-z)\right].$$

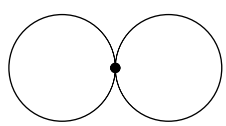
The numbers 8 and 2 are called the symmetry factors of the diagrams. The symmetry factor is given by a factor 4! for each vertex in the diagram. One then divides by the number of ways one obtains from Wick's theorem the same diagram (3 and 12 in the above example).

Alternatively, one can compute the symmetry factor as follows: S is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed.

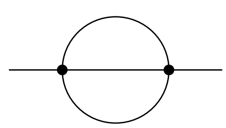
Examples:



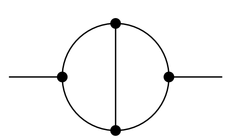
$S = 2,$



$S = 2\cdot 2\cdot 2 = 8,$

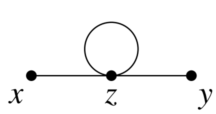


$S = 3! = 6,$



$S = 2.$

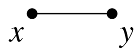
It is common practice to let a Feynman diagram represent all possible contractions leading to this diagram with the appropriate symmetry factor included. Therefore



$$= (-i\lambda)\frac{1}{2}\int d^4z D_F(x-z)D_F(y-z)D_F(z-z)$$

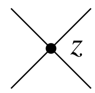
We can now formulate the **Feynman rules for ϕ^4-theory in position space**:

1. For each propagator,



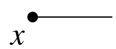
$$= D_F(x-y).$$

2. For each vertex,



$$= (-i\lambda)\int d^4z.$$

3. For each external point,



$$= 1.$$

4. Divivde by the symmetry factor S.

In practice one works in momentum space. As an example we consider the following Fourier transform

$$\begin{aligned}
&\int d^4x e^{ipx}\int d^4y e^{iqy}(-i\lambda)\frac{1}{2}\int d^4z D_F(x-z)D_F(y-z)D_F(z-z) = \\
&\quad (-i\lambda)\frac{1}{2}\int d^4x\int d^4y\int d^4z e^{i(px+qy)}\int\frac{d^4p_1}{(2\pi)^4}e^{-ip_1(x-z)}\int\frac{d^4p_2}{(2\pi)^4}e^{-ip_2(y-z)}\int\frac{d^4p_3}{(2\pi)^4}e^{-ip_3(z-z)} \\
&\quad \times\tilde{D}_F(p_1)\tilde{D}_F(p_2)\tilde{D}_F(p_3) \\
&= (-i\lambda)\frac{1}{2}\int d^4p_1\int d^4p_2\int d^4p_3\delta^4(p_1-p)\delta^4(p_2-q)\delta^4(p_1+p_2-p_3+p_3) \\
&\quad \times\tilde{D}_F(p_1)\tilde{D}_F(p_2)\tilde{D}_F(p_3) \\
&= (-i\lambda)\frac{1}{2}(2\pi)^4\delta^4(p+q)\tilde{D}_F(p)\tilde{D}_F(q)\int\frac{d^4p_3}{(2\pi)^4}\tilde{D}_F(p_3)
\end{aligned}$$

We see that we obtain one delta-function, enforcing energy-momentum conservation of the incoming and outgoing momenta. Further at each vertex, energy and momentum are conserved. Finally we have to integrate over any unrestricted four-momentum. We use the notation

$$G^n_{\text{with vacuum graphs}}(x_1,...,x_n) = \left\langle 0\left|T\left\{\hat{\phi}_I(x_1)...\hat{\phi}_I(x_n)\right\}\right|0\right\rangle$$

for the Green function with n external legs. The sub-script "with vacuum graphs" will be explained at the end of this section. We define the Fourier transform of $G^n_{\text{with vacuum graphs}}(x_1,...,x_n)$ by

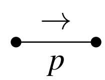
$$\begin{aligned}
&G^n_{\text{with vacuum graphs}}(x_1,...,x_n) = \\
&\quad\int\frac{d^4p_1}{(2\pi)^4}...\frac{d^4p_n}{(2\pi)^4}e^{-i\sum p_jx_j}(2\pi)^4\delta(p_1+...+p_n)\tilde{G}^n_{\text{with vacuum graphs}}(p_1,...,p_n)
\end{aligned}$$

Remark: This is not the standard Fourier transform, as we have factored out the term

$$(2\pi)^4 \delta(p_1 + \ldots + p_n).$$

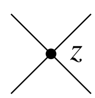
This term expresses overall momentum conservation and is always there, therefore it is convenient to factor it out. With these conventions we can now state the **Feynman rules for ϕ^4-theory in momentum space** to compute $\tilde{G}^n_{\text{with vacuum graphs}}(p_1, \ldots, p_n)$:

1. For each propagator,



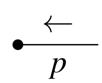
$$= \frac{i}{p^2 - m^2 + i\varepsilon}.$$

2. For each vertex,



$$= (-i\lambda).$$

3. For each external point,



$$= 1.$$

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined momentum;

$$\int \frac{d^4 p}{(2\pi)^4}$$

6. Divivde by the symmetry factor S.

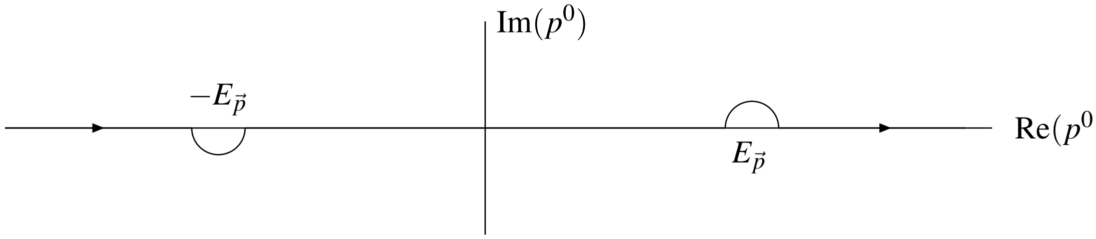
Remark: In the original formula in position space we had

$$\lim_{T \to \infty(1-i\varepsilon)} \int_{-T}^{T} dt \int d^3 z$$

which we wrote for simplicity as

$$\int d^4 z.$$

Still we have to keep in mind that for the time-component we have to integrate over a slightly distorted contour. When going to momentum space this is equivalent to integrate over the p^0-component along the following contour:



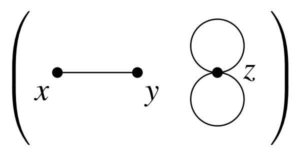
This is just the Feynman prescription.

Let us go back to our original formula

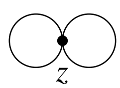
$$\left\langle \Omega \middle| T\hat{\phi}_H(x)\hat{\phi}_H(y) \middle| \Omega \right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\left\langle 0 \middle| T\left\{\hat{\phi}_I(x)\hat{\phi}_I(y)\exp\left[-i\int_{-T}^{T} dt\,\hat{H}_I(t)\right]\right\} \middle| 0 \right\rangle}{\left\langle 0 \middle| T\left\{\exp\left[-i\int_{-T}^{T} dt\,\hat{H}_I(t)\right]\right\} \middle| 0 \right\rangle}$$

In the numerator we had contributions like

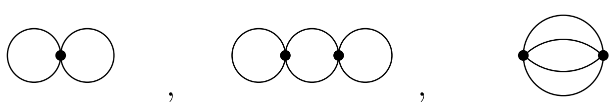
$$(-i\lambda)\int d^4z \frac{1}{8} D_F(x-y)D_F(z-z)D_F(z-z) =$$



The piece



is called a vacuum diagram. It is not connected to the external points x and y. Examples for vacuum diagrams are:



It is clear that the denominator in the formula for $\left\langle \Omega \middle| T\hat{\phi}_H(x)\hat{\phi}_H(y) \middle| \Omega \right\rangle$ produces only vacuum graphs. Let us label the values of the various vacuum graphs by V_i, and the values of the pieces connected to x and y by D_j. If a diagram consists of a piece D_j and n_1 vacuum graphs of type V_1, n_2 vacuum graphs V_2, ..., then the value for the total diagram is

$$D_j \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

The factor $n_i!$ is the symmetry factor for n_i copies of V_i. Then the numerator of the formula for the two-point correlation function is

$$\sum_j \sum_{\{n_i\}} D_j \prod_i \frac{1}{n_i!}(V_i)^{n_i},$$

where $\{n_i\}$ means all ordered sets $\{n_1, n_2, ...\}$ of non-negative integers. We have the following exponentiation:

$$\begin{aligned} \sum_j \sum_{\{n_i\}} D_j \prod_i \frac{1}{n_i!} (V_i)^{n_i} &= \sum_j D_j \prod_i \sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i} \\ &= \sum_j D_j \prod_i \exp(V_i) \\ &= \left(\sum_j D_j \right) \exp\left(\sum_i V_i \right) \end{aligned}$$

On the other hand we find for the denominator

$$\left\langle 0 \left| T \left\{ \exp\left[-i \int\limits_{-T}^{T} dt\, \hat{H}_I(t) \right] \right\} \right| 0 \right\rangle = \exp\left(\sum_i V_i \right).$$

Therefore the exponential cancels between numerator and denominator and we finally obtain

$$\left\langle \Omega \left| T \hat{\phi}_H(x) \hat{\phi}_H(y) \right| \Omega \right\rangle = \text{sum of all diagrams without any vacuum graphs.}$$

5.8 Summary: Feynman rules for ϕ^4-theory in momentum space

ϕ^4-theory is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \right) \left(\partial^\mu \phi \right) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$

We are interested in

$$G^n(x_1, ..., x_n) = \left\langle \Omega \left| T \left\{ \hat{\phi}_H(x_1) ... \hat{\phi}_H(x_n) \right\} \right| \Omega \right\rangle$$

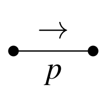
and it's Fourier transform defined through

$$G^n(x_1, ..., x_n) = \int \frac{d^4 p_1}{(2\pi)^4} ... \frac{d^4 p_n}{(2\pi)^4} e^{-i \sum p_j x_j} (2\pi)^4 \delta(p_1 + ... + p_n) \tilde{G}^n(p_1, ..., p_n)$$

(Previously we had considered $G^n_{\text{with vacuum graphs}}$, as discussed at the end of the last section, G^n and $G^n_{\text{with vacuum graphs}}$ differ by vacuum graphs.)

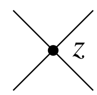
We have the following Feynman rules in momentum space: $\tilde{G}^n(p_1, ..., p_n)$ is given as the sum of all diagrams without any vacuum graphs with the following rules:

1. For each propagator,



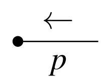
$$= \frac{i}{p^2 - m^2 + i\varepsilon}.$$

2. For each vertex,



$$= \quad -i\lambda.$$

3. For each external point,



$$= \quad 1.$$

4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined momentum;

$$\int \frac{d^4 p}{(2\pi)^4}$$

6. Divivde by the symmetry factor S.

6 Cross sections and decay rates

We have learned how to compute an abstract quantity, – the n-point correlation function –, through Feynman diagrams. We now want to relate this quantity to quantities, which can be measured: Cross sections and decay rates.

Elementary particle experiments are often scattering experiments: One collides two beams of particles with well-defined momenta, and observes what comes out. The likelihood of any particular final state can be expressed in terms of the cross section. This is a quantity intrinsic to the colliding particles and therefore allows comparison of two different experiments with different beam sizes and intensities.

The cross section is defined as follows: Consider a bunch of particles of type A with density ρ_A (particles per unit volume). Denote the length of the bunch by l_A. Assume further that the bunch is moving in the positive z-direction. Consider a second bunch of particles of type B with number density ρ_B and bunch length l_B, moving along the negative z-direction. Denote further by F the cross-sectional area common to the two bunches. We expect the total number of scattering events to be proportional to ρ_A, l_A, ρ_B, l_B and F. The cross section, denoted by σ, is just the total number of events divided by all these quantities:

$$\sigma = \frac{\text{Number of scattering events}}{\rho_A l_A \rho_B l_B F}.$$

Obviously, this definition is symmetric between A and B, therefore we could have worked in any other reference frame. In the early days of particle physics, scattering experiments were often

performed as **fixed-target experiments**. For fixed-target experiments it is common practice to use the laboratory frame **lab frame**. In the lab frame one sort of particles is at rest, say A, while the other sort moves with velocity v towards the target. Modern scattering experiments are **collider experiments**. Here the **centre-of-mass frame** is more convenient.
The cross section has units of area.
In real beams, ρ_A and ρ_B are not constant, the particle density is generally larger at the center of the beam than at the edges and one obtains

$$\text{Number of scattering events} = \sigma\, l_A l_B \int d^2x\, \rho_A(x)\rho_B(x).$$

If the densities are constant, we simply have

$$\text{Number of scattering events} = \frac{\sigma N_A N_B}{F},$$

where N_A and N_B are the total number of A and B particles. N_A and N_B are given by

$$N_A = \rho_A l_A F, \qquad N_B = \rho_B l_B F.$$

Cross sections for many different processes may be relevant to a single scattering experiment. In e^+e^- collisions, for example, one can measure the cross section for the production of $\mu^+\mu^-$, $\tau^+\tau^-$, $\mu^+\mu^-\gamma$, etc.. If one is interested in the cross section for the production of a specific final state (for example exactly one μ^+ and one μ^-) one speaks about an **exclusive cross section**, If on the other hand several final states may contribute to the cross section, one speaks about an **inclusive cross section**. An example for a typical inclusive cross section would be one μ^+ plus anything else.

Usually we wish to measure not only what the final-state particles are, but also the momenta with which they come out. For that we define the **differential cross section**

$$\frac{d\sigma}{d^3p_1...d^3p_n}$$

which is the quantity when integrated over any small $d^3p_1...d^3p_n$ gives the cross section for scattering into that region of final-state phase space. Note that the various final-state momenta are not all independent: Four components will always be constrained by momentum conservation. In the simplest case of only two final-state particles this leaves only two unconstrained variables, usually taken to be the angles θ and ϕ of the momentum of one of the particles. Integrating

$$\frac{d\sigma}{d^3p_1 d^3p_2}$$

over the four constrained momentum components then leaves us with the usual differential cross section

$$\frac{d\sigma}{d\Omega}.$$

A second measurable quantity is the decay rate of an unstable particle. The decay rate Γ of an unstable particle A, which is assumed to be at rest, is defined as follows:

$$\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of } A \text{ particles present}}.$$

The lifetime τ is the reciprocal of the sum of the decay widths into all possible final states. The particle's half-life time is $\tau \cdot \ln 2$.

In non-relativistic quantum mechanics, an unstable atomic state shows up in scattering experiments as a resonance. Near the resonance energy E_0, the scattering amplitude is given by the Breit-Wigner-formula

$$f(E) \sim \frac{1}{E - E_0 + i\Gamma/2}$$

The cross section therefore has a peak of the form

$$\sigma \sim \frac{1}{(E - E_0)^2 + \Gamma^2/4}$$

The width of the resonance peak is equal to the decay rate of the unstable state. The Breit-Wigner formula also applies to relativistic quantum mechanics. In an amplitude involving an unstable particle of momentum p and mass m we have

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\vec{p}} \left(p^0 - E_{\vec{p}} + i\left(m/E_{\vec{p}}\right)\Gamma/2\right)}.$$

Remark: A correct treatment of unstable particles in higher orders in perturbation theory is a topic of current interest.

6.1 The S-matrix

To calculate the cross section we set up wavepackets representing the initial-state particles, evolve this initial state for a very long time with the time-evolution operator $\exp(-i\hat{H}t)$ of the full interacting field theory and overlap the resulting final state with wavepackets representing some desired set of final-state particles.

A wavepacket is given by

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k}) \left|\vec{k}\right\rangle,$$

where $|\vec{k}\rangle$ is a one-particle state of momentum $\vec{k}$ in the full interacting theory and the weight function $\phi(\vec{k})$ is peaked around $\vec{p}$.

We recall that we chose as normalisation for the one-particle states

$$\langle\vec{p}|\vec{q}\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q}).$$

If we further require that

$$\int \frac{d^3k}{(2\pi)^3}\left|\phi\left(\vec{k}\right)\right|^2 = 1,$$

then we have

$$\langle\phi|\phi\rangle = 1.$$

In a typical scattering experiment one scatters two initial-state particles and one observes, say, n final-state particles. This leads us to the definition of **in- and out-states**. The in-state $|\phi_A\phi_B\rangle$ consists of the two particles A and B in the initial state, the out-state $|\phi_1\phi_2...\phi_n\rangle$ of the n particles in the final state. In the far past ($t\to-\infty$), the in-state is described by two wave-packets, corresponding to two well-sepatated single particle states and we may describe the in-state in the far past as a free state. As time proceeds, the two particles start to interact and scatter. The situation for the out-state is analogous: In the far future ($t\to\infty$), the out-state is described by n wave-packets, corresponding to n well-sepatated single particle states and we may describe in the far future the out-state as a free state. States which are described in the far past or far future as free states are called **asymptotic states**. We will denote a state, which corresponds at $t\to-\infty$ to a free n-particle state as an in-state

$$|\phi_1\phi_2...\phi_n\rangle_{\text{in}} = |\phi_1\phi_2...\phi_n;t_0=-\infty\rangle_H$$

and a state, which corresponds at $t\to\infty$ to a free n-particle state as an out-state

$$|\phi_1\phi_2...\phi_n\rangle_{\text{out}} = |\phi_1\phi_2...\phi_n;t_0=\infty\rangle_H.$$

The notation on the right-hand-side indicates that the Heisenberg state $|\phi_1\phi_2...\phi_n;t_0\rangle_H$ coincides at $t_0\in\{\pm\infty\}$ with a free n-particle state. The probability we wish to compute is

$$P = \left|{}_{\text{out}}\langle\phi_1\phi_2...\phi_n|\phi_A\phi_B\rangle_{\text{in}}\right|^2$$

The in-state $|\phi_A\phi_B\rangle_{\text{in}}$ has a nice interpretation as two localised wavepackets in the far past, but due to the evolution with the full interacting Hamiltonian in $e^{-i\hat{H}t}$, the wave packets will in general not stay intact as time proceeds. In a similar way, the out-state ${}_{\text{out}}\langle\phi_1\phi_2...\phi_n|$ has a nice interpretation as n localised wavepackets in the far future, however evolving this state backwards in time with the full interacting Hamiltonian will in general not leave the wavepackets intact. What is important is that the in- and out-states may have a non-zero overlap, which we wish to compute.

Let us first consider the in-states: Consider particle A at rest and particle B along the z-axis. It is important to take the transverse displacement into account. If the wave packet of particle B

is not centered at $x = y = 0$ in the transverse plane, but at $\vec{b}$, this yields a factor $\exp(-i\vec{b}\cdot\vec{k}_B)$. $|\vec{b}|$ is called the impact parameter.

$$|\phi_A\phi_B\rangle_{\text{in}} \quad = \quad \int \frac{d^3k_A}{(2\pi)^3}\int \frac{d^3k_B}{(2\pi)^3}\frac{\phi_A(\vec{k}_A)\phi_B(\vec{k}_B)}{\sqrt{(2E_{\vec{k}_A})(2E_{\vec{k}_B})}}e^{-i\vec{b}\cdot\vec{k}_B}\left|\vec{k}_A\vec{k}_B\right\rangle_{\text{in}}.$$

The wave packet of particle A is centered around p_A in momentum space, the one of particle B is centered around p_B. In a similar way we could write for the out-states

$$_{\text{out}}\langle\phi_1\phi_2...\phi_n| \quad = \quad \left(\prod_f \int \frac{d^3q_f}{(2\pi)^3}\frac{\phi_f(\vec{q}_f)}{\sqrt{2E_{\vec{q}_f}}}\right)\,_{\text{out}}\langle\vec{q}_1\vec{q}_2...\vec{q}_n|\,.$$

Again, the wave packet for particle f is centered around p_f in momentum space. However, in order to avoid some technical difficulties it is simpler to work for the out-state not with wavepackets but to use a state with definite momenta

$$_{\text{out}}\langle\vec{p}_1\vec{p}_2...\vec{p}_n|$$

and to put in various normalisation factors (essentially a factor $1/(2E_{\vec{p}_f})$ for every final-state particle) by hand.

Let us recall the connection with the Schrödinger picture:

$$|\phi\rangle_H \quad = \quad e^{i\hat{H}t}\,|\phi,t\rangle_S\,.$$

If we would like to indicate that a state has specific properties at a particular time t_0 (e.g. for the case at hand, if a state corresponds at $t = t_0$ to a n-particle state with momenta $\vec{p}_1$, $\vec{p}_2$, ..., $\vec{p}_n$) we will put an extra argument and write

$$|\phi;t_0\rangle_H \quad = \quad e^{i\hat{H}t}\,|\phi,t;t_0\rangle_S\,.$$

We have

$$\begin{aligned}
_{\text{out}}\langle\vec{p}_1\vec{p}_2...\vec{p}_n|\vec{k}_A\vec{k}_B\rangle_{\text{in}} \quad &= \quad \lim_{T\to\infty}\,_H\langle\vec{p}_1\vec{p}_2...\vec{p}_n;T|\vec{k}_A\vec{k}_B;-T\rangle_H \\
&= \quad \lim_{T\to\infty}\,_S\left\langle\vec{p}_1\vec{p}_2...\vec{p}_n,T;T\left|e^{-i\hat{H}(2T)}\right|\vec{k}_A\vec{k}_B,-T;-T\right\rangle_S
\end{aligned}$$

The state $|\vec{p}_1\vec{p}_2...\vec{p}_n,t_0;t_0\rangle_S$ corresponds at $t = t_0$ to a n-particle state with momenta $\vec{p}_1$, $\vec{p}_2$, ..., $\vec{p}_n$ and we will simply write $|\vec{p}_1\vec{p}_2...\vec{p}_n\rangle$. Thus

$$_{\text{out}}\langle\vec{p}_1\vec{p}_2...\vec{p}_n|\vec{k}_A\vec{k}_B\rangle_{\text{in}} \quad = \quad \lim_{T\to\infty}\left\langle\vec{p}_1\vec{p}_2...\vec{p}_n\left|e^{-i\hat{H}(2T)}\right|\vec{k}_A\vec{k}_B\right\rangle.$$

We define the S-matrix by

$$_{\text{out}}\langle\vec{p}_1\vec{p}_2...\vec{p}_n|\vec{k}_A\vec{k}_B\rangle_{\text{in}} \quad = \quad \left\langle\vec{p}_1\vec{p}_2...\vec{p}_n\left|\hat{S}\right|\vec{k}_A\vec{k}_B\right\rangle.$$

If the particles do not interact at all, $\hat{S}$ is the identity operator. Even if the theory contains interactions, the particles have some probability of missing each other. To isolate the interesting part of the $\hat{S}$-matrix – that is, the part due to interactions – we define the $\hat{T}$-matrix by

$$\hat{S} \;=\; \mathbf{1} + i(2\pi)^4\delta^4\left(k_A + k_B - \sum_f p_f\right)\hat{T}.$$

Of special interest are the matrix elements of $\hat{T}$:

$$i\mathcal{A}\left(k_A k_B \to p_1 p_2 \ldots p_n\right) \;=\; \langle \vec{p}_1 \vec{p}_2 \ldots \vec{p}_n | i\hat{T} | \vec{k}_A \vec{k}_B \rangle$$

The probability we want to compute is now

$$P \;=\; \left(\prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}_f}}\right) \left|{}_{\rm out}\langle \vec{p}_1 \vec{p}_2 \ldots \vec{p}_n | \phi_A \phi_B \rangle_{\rm in}\right|^2.$$

The bracket in front gives the correct normalisation factor for the out-state. The in-state still depends on the impact parameter $\vec{b}$. To obtain the cross section we have to integrate over all impact parameters:

$$\sigma \;=\; \int d^2 b\, P(\vec{b}).$$

Therefore

$$\frac{d\sigma}{\frac{d^3 p_1}{(2\pi)^3}\frac{d^3 p_2}{(2\pi)^3}\cdots} \;=\; \left(\prod_f \frac{1}{2E_{\vec{p}_f}}\right) \int d^2 b \left(\prod_{i=A,B} \int \frac{d^3 k_i}{(2\pi)^3} \frac{\phi_i(\vec{k}_i)}{\sqrt{2E_{\vec{k}_i}}} \int \frac{d^3 k_i'}{(2\pi)^3} \frac{\phi_i^*(\vec{k}_i')}{\sqrt{2E_{\vec{k}_i'}}}\right)$$
$$e^{-i\vec{b}\cdot(\vec{k}_B - \vec{k}_B')}\, {}_{\rm out}\langle \vec{p}_1 \vec{p}_2 \ldots \vec{p}_n | k_A k_B \rangle_{\rm in}\; {}_{\rm out}\langle \vec{p}_1 \vec{p}_2 \ldots \vec{p}_n | k_A' k_B' \rangle_{\rm in}^*.$$

If we are not interested in the trivial case where no scattering case takes place, we can drop the identity from the S-matrix and write

$$\begin{aligned}
{}_{\rm out}\langle \vec{p}_1 \vec{p}_2 \ldots \vec{p}_n | k_A k_B \rangle_{\rm in} &= (2\pi)^4 \delta^4(k_A + k_B - \sum_f p_f) i\mathcal{A}\left(k_A k_B \to p_1 p_2 \ldots p_n\right), \\
{}_{\rm out}\langle \vec{p}_1 \vec{p}_2 \ldots \vec{p}_n | k_A' k_B' \rangle_{\rm in} &= (2\pi)^4 \delta^4(k_A' + k_B' - \sum_f p_f) i\mathcal{A}\left(k_A' k_B' \to p_1 p_2 \ldots p_n\right).
\end{aligned}$$

We can use the second of these delta functions, together with the $\delta^2(k_B^\perp - k_B^{\perp\prime})$ to perform all six k' integrals. We have

$$\begin{aligned}
&\int \frac{d^3 k_A'}{(2\pi)^3} \int \frac{d^3 k_B'}{(2\pi)^3} (2\pi)^2 \delta^2(k_B^\perp - k_B^{\perp\prime}) (2\pi)^4 \delta^4(k_A' + k_B' - k_A - k_B) = \\
&= \int dk_A^{z\prime} \int dk_B^{z\prime} \delta(E_A' + E_B' - E_A - E_B) \delta(k_A^{z\prime} + k_B^{z\prime} - k_A^z - k_B^z) \\
&= \int dk_A^{z\prime}\, \delta(E_A' + E_B' - E_A - E_B)\big|_{k_B^{z\prime} = k_A^z + k_B^z - k_A^{z\prime}} \\
&= \frac{1}{\left|\frac{\partial E_A}{\partial k_A^z} - \frac{\partial E_B}{\partial k_B^z}\right|} = \frac{1}{|v_A - v_B|}.
\end{aligned}$$

The difference $|v_A - v_B|$ is the relative velocity of the beams in the lab frame. The initial wave packets are localised in momentum space, centered around p_A and p_B. We evaluate all factors, which are smooth functions of k_A and k_B at p_A and p_B. We obtain

$$\frac{d\sigma}{\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\cdots} = \left(\prod_f \frac{1}{2E_{\vec{p}_f}}\right) \frac{|\mathcal{A}(p_A p_B \to p_1 p_2 ... p_n)|^2}{2E_A 2E_B |v_A - v_B|}$$
$$\int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \left|\phi_A(\vec{k}_A)\right|^2 \left|\phi_B(\vec{k}_B)\right|^2 (2\pi)^4 \delta^4(k_A + k_B - \sum_f p_f).$$

We can simplify this formula further by noting that real detectors cannot resolve small variations of the incoming and outcoming momenta. Therefore we can approximate $\delta(k_A + k_B - \sum p_f)$ by $\delta(p_A + p_B - \sum p_f)$ and we obtain

$$\frac{d\sigma}{\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\cdots} = \left(\prod_f \frac{1}{2E_{\vec{p}_f}}\right) \frac{|\mathcal{A}(p_A p_B \to p_1 p_2 ... p_n)|^2}{2E_A 2E_B |v_A - v_B|} (2\pi)^4 \delta^4(p_A + p_B - \sum_f p_f).$$

Note that all dependence on the shape of the wave packets has disappeared.

By a similar reasoning we find for the decay rate

$$\frac{d\Gamma}{\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\cdots} = \left(\prod_f \frac{1}{2E_{\vec{p}_f}}\right) \frac{|\mathcal{A}(p_A \to p_1 p_2 ... p_n)|^2}{2m_A} (2\pi)^4 \delta^4(p_A - \sum p_f).$$

To obtain the total cross section or the total decay rate we integrate over all final states. If the final state contains n identical particles, we have to be careful not to count the same final state more then once. One possibility is to impose a strict ordering on the energies of the final state particles

$$E_1 > E_2 > ... > E_n,$$

a different (and in practice the preferred option) is to integrate without any restriction and to divide the result by a factor of $1/n!$. Therefore

$$\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|}$$
$$\frac{1}{n!} \int \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_{\vec{p}_f}}\right) (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \left|\mathcal{A}(p_A p_B \to p_1 p_2 ... p_n)\right|^2,$$
$$\Gamma = \frac{1}{2m_A} \frac{1}{n!} \int \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_{\vec{p}_f}}\right) (2\pi)^4 \delta^4(p_A - \sum p_f) \left|\mathcal{A}(p_A \to p_1 p_2 ... p_n)\right|^2.$$

We denote the Lorentz-invariant phase-space element by

$$d\phi(Q;p_1,p_2,...,p_n) = \frac{1}{n!}\left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_{\vec{p}_f}}\right)(2\pi)^4\delta^4(Q-\sum p_f)$$
$$= \frac{1}{n!}\left(\prod_f \frac{d^4p_f}{(2\pi)^4}(2\pi)\delta(p_f^2-m^2)\theta(E_{\vec{p}_f})\right)(2\pi)^4\delta^4(Q-\sum p_f).$$

The flux factor $1/(4E_A E_B|v_a - v_B|)$ is not Lorentz invariant. For beams along the z-axis the flux factor is invariant under boosts along the z-axis and we have for beams along the z-axis

$$\frac{1}{2E_A 2E_B|v_A-v_B|} = \frac{1}{2K(Q^2)},$$

where

$$2K(Q^2) = 2\sqrt{\left(Q^2-(m_1+m_2)^2\right)\left(Q^2-(m_1-m_2)^2\right)}$$
$$= 2Q^2 \quad \text{for massless particles,}$$

and $Q^2 = (p_A + p_B)^2$.

6.2 Properties of the S-matrix

In this paragraph we will denote a generic in-state by

$$|i\rangle_{\text{in}} = \left|\vec{k}_1\vec{k}_2...\vec{k}_m\right\rangle_{\text{in}},$$

and a generic out-state by

$$|f\rangle_{\text{out}} = |\vec{p}_1\vec{p}_2...\vec{p}_n\rangle_{\text{out}}.$$

We defined the S-matrix by

$$_{\text{out}}\langle f|i\rangle_{\text{in}} = \left\langle f\left|\hat{S}\right|i\right\rangle = \lim_{T\to\infty}\left\langle f\left|e^{-i\hat{H}(2T)}\right|i\right\rangle,$$

and therefore

$$\hat{S} = \lim_{T\to\infty} e^{-i\hat{H}(2T)}.$$

Since $\hat{H}$ is hermitian, the S-matrix is unitary:

$$\hat{S}^{-1} = \hat{S}^\dagger.$$

Let us further denote

$$S_{fi} = \left\langle f\left|\hat{S}\right|i\right\rangle.$$

The Hilbert spaces of the in- and out-states are isomorphic Fock spaces. Thus we may express any in-state $|i\rangle_{\text{in}}$ as a linear combination of out-states $|f\rangle_{\text{out}}$ and vice versa. The relation is provided by the S-matrix:

$$|i\rangle_{\text{in}} = \sum_f S_{fi}|f\rangle_{\text{out}}, \qquad |f\rangle_{\text{out}} = \sum_i \left(S_{fi}\right)^\dagger |i\rangle_{\text{in}}.$$

6.3 Relation between invariant matrix elements and Feynman diagrams

Here we sketch only a heuristic argument for the relation of the S-matrix to Feynman diagrams and postpone the proof to later. The proof is based on the reduction formula of Lehmann, Symanzik and Zimmermann. The S-matrix was defined by

$$ {}_{\text{out}}\langle \vec{p}_1\vec{p}_2...\vec{p}_n|\vec{p}_A\vec{p}_B\rangle_{\text{in}} \;=\; \left\langle \vec{p}_1\vec{p}_2...\vec{p}_n \left|\hat{S}\right| \vec{p}_A\vec{p}_B \right\rangle \;=\; \lim_{T\to\infty} \left\langle \vec{p}_1\vec{p}_2...\vec{p}_n \left| e^{-i\hat{H}(2T)} \right| \vec{p}_A\vec{p}_B \right\rangle $$

To compute this quantity we would like to replace the states $|\vec{p}_A\vec{p}_B\rangle$ and $|\vec{p}_1\vec{p}_2...\vec{p}_n\rangle$, which are states in the full interacting theory, with their counterparts in the free theory with Hamiltonian $\hat{H}_0$. We will denote the corresponding states in the free theory by $|\vec{p}_A\vec{p}_B\rangle_0$ and $|\vec{p}_1\vec{p}_2...\vec{p}_n\rangle_0$. For the vacuum state we already found such a formula

$$ |\Omega\rangle \;=\; \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0^{\text{full}}T} \langle\Omega|0\rangle \right)^{-1} e^{-i\hat{H}T} |0\rangle. $$

Now we look for a relation of the form

$$ |\vec{p}_A\vec{p}_B\rangle \;\sim\; \lim_{T\to\infty(1-i\varepsilon)} e^{-i\hat{H}T} \, |\vec{p}_A\vec{p}_B\rangle_0 \, , $$

where we have omitted the prefactor. We change to the interaction picture:

$$ \begin{aligned} |\vec{p}_A\vec{p}_B\rangle_{\text{in}} &= e^{i\hat{H}(-T-t_0)} \, |\vec{p}_A\vec{p}_B,-T\rangle_S \;=\; e^{i\hat{H}(-T-t_0)} e^{-i\hat{H}_0(-T-t_0)} \, |\vec{p}_A\vec{p}_B,-T\rangle_I \\ &= \hat{U}^\dagger(-T,t_0) \, |\vec{p}_A\vec{p}_B,-T\rangle_I \, , \\ |\vec{p}_1\vec{p}_2...\vec{p}_n\rangle_{\text{out}} &= e^{i\hat{H}(T-t_0)} \, |\vec{p}_1\vec{p}_2...\vec{p}_n,T\rangle_S \;=\; e^{i\hat{H}(T-t_0)} e^{-i\hat{H}_0(T-t_0)} \, |\vec{p}_1\vec{p}_2...\vec{p}_n,T\rangle_I \\ &= \hat{U}^\dagger(T,t_0) \, |\vec{p}_1\vec{p}_2...\vec{p}_n,T\rangle_I \, , \end{aligned} $$

and therefore

$$ {}_{\text{out}}\langle \vec{p}_1\vec{p}_2...\vec{p}_n,T|\vec{p}_A\vec{p}_B\rangle_{\text{in}} \;=\; \lim_{T\to\infty(1-i\varepsilon)} \; {}_I\left\langle \vec{p}_1\vec{p}_2...\vec{p}_n \left| \hat{U}(T,t_0)\,\hat{U}^\dagger(-T,t_0) \right| \vec{p}_A\vec{p}_B,-T \right\rangle_I $$

We have

$$ \hat{U}(T,t_0)\,\hat{U}^\dagger(-T,t_0) \;=\; \hat{U}(T,-T) \;=\; T\left\{ \exp\left[-i\int_{-T}^{T} dt\, \hat{H}_I(t) \right] \right\}. $$

The T in front of the curly brackets denotes the time-ordered product. Following the same arguments we used to derive the relation between $|\Omega\rangle$ and $|0\rangle$, one arrives at

$$ {}_{\text{out}}\langle \vec{p}_1\vec{p}_2...\vec{p}_n|\vec{p}_A\vec{p}_B\rangle_{\text{in}} \;\sim\; \lim_{T\to\infty(1-i\varepsilon)} {}_0\langle \vec{p}_1\vec{p}_2...\vec{p}_n| T\exp\left[-i\int_{-T}^{T} dt\hat{H}_I(t) \right] |\vec{p}_A\vec{p}_B\rangle_0. $$

Here we neglected all proportionality factors. We will do a more careful derivation later on, including the proportionality factors.

It can be shown that the non-trivial part of the S-matrix can be computed as follows

$$\left\langle \vec{p}_1 \vec{p}_2 ... \vec{p}_n \left| i\hat{T} \right| \vec{p}_A \vec{p}_B \right\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left({}_0\langle \vec{p}_1 \vec{p}_2 ... | T \exp \left[-i \int\limits_{-T}^{T} dt H_I(t) \right] | \vec{p}_A \vec{p}_B \rangle_0 \right) \Bigg|_{\substack{\text{connected} \\ \text{amputated}}},$$

A Feynman diagram is called **connected**, if it does not have disjoint pieces.

A Feynman diagram is called **amputated**, if for all external lines, the corresponding propagator is removed.

6.4 Final formula

Let us now summarise and present the final "master"-formula for the computation of cross sections and decay rates. Up to now we only considered scalar particles. There are a few minor modifications if the involved particles have additional degrees of freedom, like spin or colour degrees of freedom. We haven't treated spin and other degrees of freedom in detail yet, but later we will see that

$$|\mathcal{A}|^2$$

involves a sum over all spins and other degrees of freedom. This is fine for final-state particles, but for initial-state particles we have to average over these degrees of freedom. If the initial-state particles have spin, this brings an additional factor of

$$\frac{1}{n_{\text{spin}}(A) n_{\text{spin}}(B)}$$

for scattering reactions and a factor

$$\frac{1}{n_{\text{spin}}(A)}$$

for the decay rate. The quantity $n_{\text{spin}}(A)$ denotes the number of spin degrees of freedom of particle A. Similar, if the initial-state particles have colour charges (i.e. are quarks or gluons), this brings an additional factor of

$$\frac{1}{n_{\text{colour}}(A) n_{\text{colour}}(B)}$$

for scattering reactions, where $n_{\text{colour}}(A)$ denotes the number of colour degrees of freedom of particle A.

To calculate the cross section at an collider with no initial-state hadrons (e.g. an electron-positron collider):

$$\sigma = \frac{1}{2K(Q^2)} \frac{1}{n_{\text{spin}}(A) n_{\text{spin}}(B)} \int d\phi\left(p_A + p_B; p_1, ..., p_n\right) \left|\mathcal{A}\left(p_A p_B \to p_1 p_2 ...\right)\right|^2,$$

where $2K(Q^2)$ is the flux factor and we have $2K(Q^2) = 2Q^2$ for massless incoming particles. $Q = \sqrt{(p_A + p_B)^2}$ is the centre-of-mass energy. For a decay rate we have

$$\Gamma = \frac{1}{2m_A} \frac{1}{n_{\text{spin}}(A)} \int d\phi\left(p_A; p_1, ..., p_n\right) \left|\mathcal{A}\left(p_A \to p_1 p_2 ...\right)\right|^2.$$

The phase-space measure is given by

$$d\phi(Q; p_1, p_2, ..., p_n) = \frac{1}{\prod_j n_j!} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_{\vec{p}_f}} \right) (2\pi)^4 \delta^4 (Q - \sum p_f),$$

if the final state contains n_j identical particles of type j. If the colliding particles are not elementary (like protons or antiprotons), we have to include the probability of finding the elementary particle A inside the proton or antiproton. If the proton has momentum p_p one usually specifies the probability of finding a parton with momentum fraction x by the parton distirbution function

$$f(x).$$

The parton has then the momentum

$$p_A = x p_p.$$

For the cross section we have to integrate over all possible momentum fractions and the formula for a hadron-hadron collider becomes

$$\sigma = \int dx_A f(x_A) \int dx_B f(x_B) \frac{1}{2K(\hat{s})} \frac{1}{n_{\text{spin}}(A) n_{\text{spin}}(B)} \frac{1}{n_{\text{colour}}(A) n_{\text{colour}}(B)}$$
$$\int d\phi\left(p_A + p_B; p_1, ..., p_n\right) \left|\mathcal{A}\left(p_A p_B \to p_1 p_2 ...\right)\right|^2.$$

$n_{\text{colour}}(A)$ and $n_{\text{colour}}(B)$ are the number of colour degrees of the initial state particles. We further have

$$\hat{s} = (p_a + p_B)^2 = \left(x_A p_{p_A} + x_B p_{p_B}\right)^2 = x_A x_B \left(p_{p_A} + p_{p_B}\right)^2.$$

The last equality holds if $p_{p_A}^2 = p_{p_B}^2 = 0$. At high energies, like the LHC experiments, masses of the order of 1 GeV do not play an essential role and the proton can be treated approximately as massless.

7 Fermions

Up to now we considered only spin zero particles. In this section we study quantum field theories with spin $1/2$ particles. But before we embark on quantisation of these theories, we first discuss solutions of the classical theory.

7.1 The Dirac equation

The Lagrange density for a (classical) Dirac field depends on four-component spinors $\psi_\alpha(x)$ ($\alpha = 1,2,3,4$) and $\bar{\psi}_\alpha(x) = \left(\psi^\dagger(x)\gamma^0\right)_\alpha$:

$$\mathcal{L}(\psi,\bar{\psi},\partial_\mu\psi) \quad = \quad i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x)$$

Here, the (4×4)-Dirac matrices satisfy the anti-commutation rules

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}, \quad \{\gamma^\mu,\gamma_5\} = 0, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

The Euler-Lagrange equations yield the Dirac equations

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) \; = \; 0, \qquad \bar{\psi}(x)\left(i\gamma^\mu \overleftarrow{\partial}_\mu + m\right) \; = \; 0.$$

For computations it is useful to have an explicit representation of the Dirac matrices. There are several widely used representations. A particular useful one is the Weyl representation of the Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here, the 4-dimensional σ^μ-matrices are defined by

$$\sigma^\mu_{A\dot{B}} = (1,-\vec{\sigma}) \qquad \bar{\sigma}^{\mu\dot{A}B} = (1,\vec{\sigma})$$

and $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ are the standard Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let us now look for plane wave solutions of the Dirac equation. We make the Ansatz

$$\psi(x) \quad = \quad \begin{cases} u(p)e^{-ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{incoming fermion,} \\ v(p)e^{+ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{outgoing anti-fermion.} \end{cases}$$

$u(p)$ describes incoming particles, $v(p)$ describes outgoing anti-particles. Similar,

$$\bar{\psi}(x) \quad = \quad \begin{cases} \bar{u}(p)e^{+ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{outgoing fermion,} \\ \bar{v}(p)e^{-ipx}, & p^0 > 0, \quad p^2 = m^2, \quad \text{incoming anti-fermion,} \end{cases}$$

where

$$\bar{u}(p) = u^\dagger(p)\gamma^0, \qquad \bar{v}(p) = v^\dagger(p)\gamma^0.$$

$\bar{u}(p)$ describes outgoing particles, $\bar{v}(p)$ describes incoming anti-particles. Then

$$\begin{aligned} (\not{p} - m)\, u(p) = 0, &\qquad (\not{p} + m)\, v(p) = 0, \\ \bar{u}(p)\, (\not{p} - m) = 0, &\qquad \bar{v}(p)\, (\not{p} + m) = 0, \end{aligned}$$

We will find that there is more then one solution for $u(p)$ (and the other spinors $\bar{u}(p)$, $v(p)$, $\bar{v}(p)$). We will label the various solutions with λ. The degeneracy is related to the additional spin degree of freedom and we find for a spin $1/2$ particles 2 solutions. We require that the two solutions satisfy the orthogonality relations

$$\begin{aligned} \bar{u}(p,\bar{\lambda})u(p,\lambda) &= 2m\delta_{\bar{\lambda}\lambda}, \\ \bar{v}(p,\bar{\lambda})v(p,\lambda) &= -2m\delta_{\bar{\lambda}\lambda}, \\ \bar{u}(p,\bar{\lambda})v(p,\lambda) &= \bar{v}(\bar{\lambda})u(\lambda) = 0, \end{aligned}$$

and the completeness relation

$$\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) = \not{p} + m, \qquad \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) = \not{p} - m.$$

7.2 Massless spinors

Let us now try to find explicit solutions for the spinors $u(p)$, $v(p)$, $\bar{u}(p)$ and $\bar{v}(p)$. The simplest case is the one of a massless fermion:

$$m = 0.$$

In this case the Dirac equation for the u- and the v-spinors are identical and it is sufficient to consider

$$\not{p}u(p) = 0, \qquad \bar{u}(p)\not{p} = 0.$$

In the Weyl representation $\not{p}$ is given by

$$\not{p} = \begin{pmatrix} 0 & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & 0 \end{pmatrix},$$

therefore the 4×4-matrix equation for $u(p)$ (or $\bar{u}(p)$) decouples into two 2×2-matrix equations. We introduce the following notation: Four-component Dirac spinors are constructed out of two **Weyl spinors** as follows:

$$u(p) = \begin{pmatrix} |p+\rangle \\ |p-\rangle \end{pmatrix} = \begin{pmatrix} |p\rangle \\ |p] \end{pmatrix} = \begin{pmatrix} p_A \\ p^{\dot{B}} \end{pmatrix} = \begin{pmatrix} u_+(p) \\ u_-(p) \end{pmatrix}.$$

Bra-spinors are given by

$$\bar{u}(p) = (\langle p-|, \langle p+|) = (\langle p|, [p|) = \left(p^A, p_{\dot{B}} \right) = (\bar{u}_-(p), \bar{u}_+(p)).$$

In the literature there exists various notations for Weyl spinors. Eq. (1) and eq. (1) show four of them and the way how to translate from one notation to another notation. By a slight abuse of notation we will in the following not distinguish between a two-component Weyl spinor and a Dirac spinor, where either the upper two components or the lower two components are zero. If we define the chiral projection operators

$$P_+ = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then (with the slight abuse of notation mentioned above)

$$u_\pm(p) = P_\pm u(p), \quad \bar{u}_\pm(p) = \bar{u}(p) P_\mp.$$

The two solutions of the Dirac equation

$$\not{p} u(p,\lambda) = 0$$

are then

$$u(p,+) = u_+(p), \quad u(p,-) = u_-(p).$$

We now have to solve

$$\begin{aligned} p_\mu \bar{\sigma}^\mu |p+\rangle = 0, \quad & p_\mu \sigma^\mu |p-\rangle = 0, \\ \langle p+| p_\mu \bar{\sigma}^\mu = 0, \quad & \langle p-| p_\mu \sigma^\mu = 0. \end{aligned}$$

It it convenient to express the four-vector $p^\mu = (p^0, p^1, p^2, p^3)$ in terms of light-cone coordinates:

$$p^+ = \frac{1}{\sqrt{2}}\left(p^0 + p^3\right), \quad p^- = \frac{1}{\sqrt{2}}\left(p^0 - p^3\right), \quad p^\perp = \frac{1}{\sqrt{2}}\left(p^1 + ip^2\right), \quad p^{\perp *} = \frac{1}{\sqrt{2}}\left(p^1 - ip^2\right).$$

Note that $p^{\perp *}$ does not involve a complex conjugation of p^1 or p^2. For null-vectors one has

$$p^{\perp *} p^\perp = p^+ p^-.$$

Then the equation for the ket-spinors becomes

$$\begin{pmatrix} p^- & -p^{\perp *} \\ -p^\perp & p^+ \end{pmatrix} |p+\rangle = 0, \quad \begin{pmatrix} p^+ & p^{\perp *} \\ p^\perp & p^- \end{pmatrix} |p-\rangle = 0,$$

and similar equations can be written down for the bra-spinors. This is a problem of linear algebra. Solutions for ket-spinors are

$$|p+\rangle = p_A = c_1 \begin{pmatrix} p^{\perp *} \\ p^- \end{pmatrix}, \quad |p-\rangle = p^{\dot{A}} = c_2 \begin{pmatrix} p^- \\ -p^\perp \end{pmatrix},$$

with some yet unspecified multiplicative constants c_1 and c_2. Solutions for bra-spinors are

$$\langle p+| = p_{\dot{A}} = c_3 \left(p^\perp, p^- \right), \qquad \langle p-| = p^A = c_4 \left(p^-, -p^{\perp *} \right),$$

with some further constants c_3 and c_4. Let us now introduce the 2-dimensional antisymmetric tensor:

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{BA} = -\varepsilon_{AB}$$

Furthermore we set

$$\varepsilon^{AB} = \varepsilon^{\dot{A}\dot{B}} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}}.$$

Note that these definitions imply

$$\varepsilon^{AC}\varepsilon_{BC} = \delta^A_B, \qquad \varepsilon^{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{C}} = \delta^{\dot{A}}_{\dot{B}}.$$

We would like to have the following relations for raising and lowering a spinor index A or $\dot{B}$:

$$\begin{aligned} p^A &= \varepsilon^{AB} p_B, \qquad & p^{\dot{A}} &= \varepsilon^{\dot{A}\dot{B}} p_{\dot{B}}, \\ p_{\dot{B}} &= p^{\dot{A}} \varepsilon_{\dot{A}\dot{B}}, \qquad & p_B &= p^A \varepsilon_{AB}. \end{aligned}$$

Note that raising an index is done by left-multiplication, whereas lowering is performed by right-multiplication. Postulating these relations implies

$$c_1 = c_4, \qquad c_2 = c_3.$$

In addition we normalise the spinors according to

$$\langle p \pm | \gamma^\mu | p \pm \rangle = 2 p^\mu.$$

This implies

$$c_1 c_3 = \frac{\sqrt{2}}{p^-}, \qquad c_2 c_4 = \frac{\sqrt{2}}{p^-}.$$

These equations determine the spinors only up to a scaling

$$p_A \to \lambda p_A, \qquad p_{\dot{A}} \to \frac{1}{\lambda} p_{\dot{A}}.$$

This scaling freedom is referred to as **little group scaling**. Keeping the scaling freedom, we define the spinors as

$$\begin{aligned} |p+\rangle = p_A &= \frac{\lambda_p 2^{\frac{1}{4}}}{\sqrt{p^-}} \begin{pmatrix} p^{\perp *} \\ p^- \end{pmatrix}, \qquad & |p-\rangle = p^{\dot{A}} &= \frac{2^{\frac{1}{4}}}{\lambda_p \sqrt{p^-}} \begin{pmatrix} p^- \\ -p^\perp \end{pmatrix}, \\ \langle p+| = p_{\dot{A}} &= \frac{2^{\frac{1}{4}}}{\lambda_p \sqrt{p^-}} \left(p^\perp, p^- \right), \qquad & \langle p-| = p^A &= \frac{\lambda_p 2^{\frac{1}{4}}}{\sqrt{p^-}} \left(p^-, -p^{\perp *} \right). \end{aligned}$$

Popular choices for λ_p are

$$\begin{aligned} \lambda_p = 1 &\quad : \quad \text{symmetric}, \\ \lambda_p = 2^{\frac{1}{4}}\sqrt{p^-} &\quad : \quad p_A \text{ linear in } p^\mu, \\ \lambda_p = \frac{1}{2^{\frac{1}{4}}\sqrt{p^-}} &\quad : \quad p_{\dot{A}} \text{ linear in } p^\mu. \end{aligned}$$

Note that all formulæ in this sub-section work not only for real momenta p^μ but also for complex momenta p^μ. This will be useful later on, where we encounter situations with complex momenta. However there is one exception: The relations $p_A^\dagger = p_{\dot{A}}$ and $p^{A\dagger} = p^{\dot{A}}$ (or equivalently $\bar{u}(p) = u(p)^\dagger \gamma^0$) encountered in previous sub-sections are valid only for real momenta $p^\mu = (p^0, p^1, p^2, p^3)$ (and $p^- > 0$). If on the other hand the components (p^0, p^1, p^2, p^3) are complex, these relations will in general not hold. In the latter case p_A and $p_{\dot{A}}$ are considered to be independent quantities. The reason, why the relations $p_A^\dagger = p_{\dot{A}}$ and $p^{A\dagger} = p^{\dot{A}}$ do not hold in the complex case lies in the definition of $p^{\perp *}$: We defined $p^{\perp *}$ as $p^{\perp *} = (p^1 - ip^2)/\sqrt{2}$, and not as $(p^1 - i(p^2)^*)/\sqrt{2}$. With the former definition $p^{\perp *}$ is a holomorphic function of p^1 and p^2. There are applications where holomorphicity is more important than nice properties under hermitian conjugation.

For $p^\mu = (p^0, p^1, p^2, p^3)$ real and $p^0 > 0$ we have the relations

$$\begin{array}{ccccc} & \langle p - | = p^A & \xleftrightarrow{\;\;\varepsilon^{AB}\;\;} & |p+\rangle = p_A & \\ \text{hermitian conj.} & \updownarrow & & \updownarrow & \text{hermitian conj.} \\ & |p-\rangle = p^{\dot{A}} & \xleftrightarrow{\;\;\varepsilon^{\dot{A}\dot{B}}\;\;} & \langle p + | = p_{\dot{A}} & \end{array}$$

7.3 Spinorproducts

Let us now make the symmetric choice $\lambda_p = 1$. Spinor products are defined by

$$\begin{aligned} \langle pq \rangle &= \langle p - | q + \rangle = p^A q_A = \frac{\sqrt{2}}{\sqrt{p^-}\sqrt{q^-}} \left(p^- q^{\perp *} - q^- p^{\perp *} \right), \\ [qp] &= \langle q + | p - \rangle = q_{\dot{A}} p^{\dot{A}} = \frac{\sqrt{2}}{\sqrt{p^-}\sqrt{q^-}} \left(p^- q^{\perp} - q^- p^{\perp} \right), \end{aligned}$$

where the last expression in each line used the choice $\lambda_p = \lambda_q = 1$. We have

$$\langle pq \rangle [qp] = 2 p \cdot q.$$

If p^μ and q^μ are real we have

$$[qp] = \langle pq \rangle^* \,\text{sign}(p^0)\,\text{sign}(q^0).$$

The spinorproducts are anti-symmetric:

$$\begin{aligned} \langle qp \rangle &= -\langle pq \rangle, \\ [pq] &= -[qp]. \end{aligned}$$

From the Schouten identity for the 2-dimensional antisymmetric tensor

$$\varepsilon_{AB}\varepsilon_{CD} + \varepsilon_{AC}\varepsilon_{DB} + \varepsilon_{AD}\varepsilon_{BC} = 0$$

one derives

$$\begin{aligned} \langle p_1 p_2 \rangle \langle p_3 p_4 \rangle &= \langle p_1 p_4 \rangle \langle p_3 p_2 \rangle + \langle p_1 p_3 \rangle \langle p_2 p_4 \rangle \\ [p_1 p_2]\,[p_3 p_4] &= [p_1 p_4]\,[p_3 p_2] + [p_1 p_3]\,[p_2 p_4] \end{aligned}$$

Fierz identity:

$$\langle p_1 + |\gamma_\mu| p_2 + \rangle \langle p_3 - |\gamma^\mu| p_4 - \rangle = 2 [p_1 p_4] \langle p_3 p_2 \rangle$$

Useful formulas in the bra-ket notation:

$$\begin{aligned} \langle p \pm |\gamma_{\mu_1} ... \gamma_{\mu_{2n+1}}| q \pm \rangle &= \langle q \mp |\gamma_{\mu_{2n+1}} ... \gamma_{\mu_1}| p \mp \rangle \\ \langle p \pm |\gamma_{\mu_1} ... \gamma_{\mu_{2n}}| q \mp \rangle &= -\langle q \pm |\gamma_{\mu_{2n}} ... \gamma_{\mu_1}| p \mp \rangle \end{aligned}$$

7.4 Massive spinors

As in the massless case, a massive spinor satisfying the Dirac equation has a two-fold degeneracy. We will label the two different eigenvectors by "+" and "-". Let p be a massive four-vector with $p^2 = m^2$, and let q be an arbitrary light-like vector. With the help of q we can construct a light-like vector $p^\flat$ associated to p:

$$p^\flat = p - \frac{p^2}{2 p \cdot q} q.$$

We define

$$\begin{aligned} u(p,+) &= \frac{1}{\langle p^\flat + | q - \rangle} (\not{p} + m) \, | q - \rangle, & v(p,-) &= \frac{1}{\langle p^\flat + | q - \rangle} (\not{p} - m) \, | q - \rangle, \\ u(p,-) &= \frac{1}{\langle p^\flat - | q + \rangle} (\not{p} + m) \, | q + \rangle, & v(p,+) &= \frac{1}{\langle p^\flat - | q + \rangle} (\not{p} - m) \, | q + \rangle. \end{aligned}$$

For the conjugate spinors we have

$$\begin{aligned} \bar{u}(p,+) &= \frac{1}{\langle q - | p^\flat + \rangle} \langle q - | \, (\not{p} + m), & \bar{v}(p,-) &= \frac{1}{\langle q - | p^\flat + \rangle} \langle q - | \, (\not{p} - m), \\ \bar{u}(p,-) &= \frac{1}{\langle q + | p^\flat - \rangle} \langle q + | \, (\not{p} + m), & \bar{v}(p,+) &= \frac{1}{\langle q + | p^\flat - \rangle} \langle q + | \, (\not{p} - m). \end{aligned}$$

These spinors satisfy the Dirac equations

$$\begin{aligned}(\not{p}-m)\,u(\lambda)=0, \qquad (\not{p}+m)\,v(\lambda)=0,\\ \bar{u}(\lambda)\,(\not{p}-m)=0, \qquad \bar{v}(\lambda)\,(\not{p}+m)=0,\end{aligned}$$

the orthogonality relations

$$\begin{aligned}\bar{u}(\bar{\lambda})u(\lambda) &= 2m\delta_{\bar{\lambda}\lambda},\\ \bar{v}(\bar{\lambda})v(\lambda) &= -2m\delta_{\bar{\lambda}\lambda},\\ \bar{u}(\bar{\lambda})v(\lambda) &= \bar{v}(\bar{\lambda})u(\lambda)=0,\end{aligned}$$

and the completeness relation

$$\sum_{\lambda} u(p,\lambda)\bar{u}(p,\lambda) \;=\; \not{p}+m, \qquad \sum_{\lambda} v(p,\lambda)\bar{v}(p,\lambda) \;=\; \not{p}-m.$$

We further have

$$\bar{u}(p,\bar{\lambda})\gamma^{\mu}u(p,\lambda)=2p^{\mu}\delta_{\bar{\lambda}\lambda}, \qquad \bar{v}(p,\bar{\lambda})\gamma^{\mu}v(p,\lambda)=2p^{\mu}\delta_{\bar{\lambda}\lambda}.$$

In the massless limit the definition reduces to

$$\begin{aligned}u(p,+)=v(p,-)=|p+\rangle, \qquad \bar{u}(p,+)=\bar{v}(p,-)=\langle p+|,\\ u(p,-)=v(p,+)=|p-\rangle, \qquad \bar{u}(p,-)=\bar{v}(p,+)=\langle p-|,\end{aligned}$$

and the spinors are independent of the reference spinors $|q+\rangle$ and $\langle q+|$.

7.5 Quantisation of fermions

We start from the Lagrangian

$$\mathcal{L}(\psi,\bar{\psi},\partial_{\mu}\psi) \;=\; i\bar{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x)-m\bar{\psi}(x)\psi(x)$$

The canonical momentum conjugate to ψ is

$$\frac{\partial\mathcal{L}}{\partial\partial_0\psi} \;=\; i\bar{\psi}\gamma^0=i\psi^{\dagger}.$$

Thus the Hamilton function is

$$H \;=\; \int d^3x\left[i\psi^{\dagger}\partial_0\psi-\mathcal{L}\right]=\int d^3x\,\bar{\psi}\left[-i\vec{\gamma}\cdot\vec{\nabla}+m\right]\psi=\int d^3x\,\psi^{\dagger}\left[-i\gamma^0\vec{\gamma}\cdot\vec{\nabla}+m\gamma^0\right]\psi.$$

Here

$$\vec{\gamma}=\left(\gamma^1,\gamma^2,\gamma^3\right), \qquad \vec{\nabla}=\left(\partial_1,\partial_2,\partial_3\right).$$

If we define

$$\vec{\alpha} = \gamma^0\vec{\gamma}, \qquad \beta = \gamma^0,$$

we obtain

$$H = \int d^3x\, \psi^\dagger \left[-i\vec{\alpha}\cdot\vec{\nabla} + m\beta\right]\psi.$$

Let us expand the field $\psi(x)$ in a set of eigenfunctions of

$$h_D = -i\vec{\alpha}\cdot\vec{\nabla} + m\beta$$

From the solution of the free Dirac equation we already know that

$$\left[-i\vec{\alpha}\cdot\vec{\nabla} + m\beta\right] u(p,\lambda)e^{-ip\cdot x} = 0.$$

Therefore $u(p,\lambda)e^{i\vec{p}\cdot\vec{x}}$ are eigenfunctions of h_D with eigenvalues $E_{\vec{p}}$. Similarly, the functions $v(p,\lambda)e^{-i\vec{p}\cdot\vec{x}}$ are eigenfunctions of h_D with eigenvalues $-E_{\vec{p}}$. These form a complete set of eigenfunctions, since for any $\vec{p}$ there are two u's and two v's, giving us four eigenvectors of the 4×4 matrix h_D. We write for the field operators in the Heisenberg picture

$$\hat{\psi}_H(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_\lambda \left(\hat{a}_{\vec{p},\lambda}u(p,\lambda)e^{-ip\cdot x} + \hat{b}^\dagger_{\vec{p},\lambda}v(p,\lambda)e^{ip\cdot x}\right),$$

$$\hat{\bar{\psi}}_H(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_\lambda \left(\hat{b}_{\vec{p},\lambda}\bar{v}(p,\lambda)e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p},\lambda}\bar{u}(p,\lambda)e^{ip\cdot x}\right).$$

The creation and annihilation operators obey the anticommutation rules

$$\left\{\hat{a}_{\vec{p},\lambda}, \hat{a}^\dagger_{\vec{q},\lambda'}\right\} = \left\{\hat{b}_{\vec{p},\lambda}, \hat{b}^\dagger_{\vec{q},\lambda'}\right\} = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta_{\lambda\lambda'}.$$

All other anti-commutators vanish:

$$\left\{\hat{a}_{\vec{p},\lambda}, \hat{a}_{\vec{q},\lambda'}\right\} = \left\{\hat{a}^\dagger_{\vec{p},\lambda}, \hat{a}^\dagger_{\vec{q},\lambda'}\right\} = \left\{\hat{b}_{\vec{p},\lambda}, \hat{b}_{\vec{q},\lambda'}\right\} = \left\{\hat{b}^\dagger_{\vec{p},\lambda}, \hat{b}^\dagger_{\vec{q},\lambda'}\right\} = 0,$$
$$\left\{\hat{a}_{\vec{p},\lambda}, \hat{b}_{\vec{q},\lambda'}\right\} = \left\{\hat{a}_{\vec{p},\lambda}, \hat{b}^\dagger_{\vec{q},\lambda'}\right\} = \left\{\hat{a}^\dagger_{\vec{p},\lambda}, \hat{b}_{\vec{q},\lambda'}\right\} = \left\{\hat{a}^\dagger_{\vec{p},\lambda}, \hat{b}^\dagger_{\vec{q},\lambda'}\right\} = 0.$$

The anticommutation relations for $\hat{\psi}_S$ and $\hat{\psi}^\dagger_S$ in the Schrödinger picture (or equal-time anticommutation relations in the Heisenberg picture) are then

$$\left\{\hat{\psi}_S(\vec{x}), \hat{\psi}^\dagger_S(\vec{y})\right\} = \delta^3(\vec{x}-\vec{y}),$$
$$\left\{\hat{\psi}_S(\vec{x}), \hat{\psi}_S(\vec{y})\right\} = \left\{\hat{\psi}^\dagger_S(\vec{x}), \hat{\psi}^\dagger_S(\vec{y})\right\} = 0.$$

The vacuum $|0\rangle$ is defined to be the state such

$$\hat{a}_{\vec{p},\lambda}|0\rangle = \hat{b}_{\vec{p},\lambda}|0\rangle = 0.$$

The Hamiltonian can be written as

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} \sum_\lambda E_{\vec{p}} \left(\hat{a}^\dagger_{\vec{p},\lambda} \hat{a}_{\vec{p},\lambda} + \hat{b}^\dagger_{\vec{p},\lambda} \hat{b}_{\vec{p},\lambda} \right),$$

where an infinite constant has been dropped. The momentum operator is

$$\hat{\vec{P}} = \int d^3x \, \hat{\psi}_S^\dagger \left(-i\vec{\nabla} \right) \hat{\psi}_S = \int \frac{d^3p}{(2\pi)^3} \sum_\lambda \vec{p} \left(\hat{a}^\dagger_{\vec{p},\lambda} \hat{a}_{\vec{p},\lambda} + \hat{b}^\dagger_{\vec{p},\lambda} \hat{b}_{\vec{p},\lambda} \right)$$

Thus both $\hat{a}^\dagger_{\vec{p},\lambda}$ and $\hat{b}^\dagger_{\vec{p},\lambda}$ create particles with energy $+E_{\vec{p}}$ and momentum $\vec{p}$. We will refer to the particles created by $\hat{a}^\dagger_{\vec{p},\lambda}$ as fermions and to those created by $\hat{b}^\dagger_{\vec{p},\lambda}$ as antifermions.

The one-particle states

$$|\vec{p},\lambda\rangle = \sqrt{2E_{\vec{p}}} \hat{a}^\dagger_{\vec{p},\lambda} |0\rangle$$

are defined so that their inner product

$$\langle \vec{p},\lambda | \vec{q},\lambda' \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \delta_{\lambda\lambda'}$$

is Lorentz invariant.

7.6 Feynman rules for fermions

To apply Wick's theorem we have to generalise the definitions of the time-ordered product and the normal product towards anticommuting operators. We make the following defintions:

$$T \hat{\psi}(x) \hat{\bar{\psi}}(y) = \begin{cases} \hat{\psi}(x)\hat{\bar{\psi}}(y) & \text{for } x^0 > y^0 \\ -\hat{\bar{\psi}}(y)\hat{\psi}(x) & \text{for } y^0 > x^0 \end{cases}$$

For the normal product we define

$$\begin{aligned} :\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{q}}: &= \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{q}}, \\ :\hat{a}_{\vec{q}} \hat{a}^\dagger_{\vec{p}}: &= -\hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{q}}, \end{aligned}$$

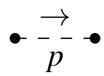
where $\hat{a}^\dagger_{\vec{p}}$ and $\hat{a}_{\vec{q}}$ are creation and annihilation operators for fermions. Therefore a minus sign occurs everytime we have to exchange two fermionic operators.

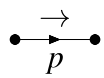
The Lagrange density for the Yukawa theory:

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Klein-Gordon}} - g\bar{\psi}\psi\phi, \\ \mathcal{L}_{\text{Dirac}} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi, \\ \mathcal{L}_{\text{Klein-Gordon}} &= \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2. \end{aligned}$$

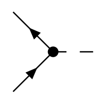
We obtain the following Feynman rules in momentum space:

1. For each propagator,

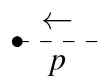
 $$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

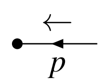
 $$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

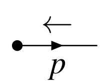
2. For each vertex,

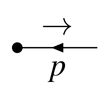
 $$= -ig$$

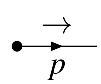
3. For each external point,

 $$= 1$$

 $$= \bar{u}(p, \lambda)$$

 $$= v(p, \lambda)$$

 $$= \bar{v}(p, \lambda)$$

 $$= u(p, \lambda)$$

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined momentum;

$$\int \frac{d^4 p}{(2\pi)^4}$$

6. Divivde by the symmetry factor S.

7. For each closed fermion loop a factor of (-1).

7.7 Rules for traces over Dirac matrices

In evaluating the amplitude squared we encounter trace over Dirac matrices:
Theorem 1:

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu = 4 g^{\mu\nu}$$

Proof:

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu + \mathrm{Tr}\,\gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathrm{Tr}\,1 = 8g^{\mu\nu}$$

Theorem 2:

$$\mathrm{Tr}\,\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n}} = g^{\mu_1\mu_2}\mathrm{Tr}\,\gamma^{\mu_3}...\gamma^{\mu_{2n}} - g^{\mu_1\mu_3}\mathrm{Tr}\,\gamma^{\mu_2}\gamma^{\mu_4}...\gamma^{\mu_{2n}} + g^{\mu_1\mu_4}\mathrm{Tr}\,\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_5}...\gamma^{\mu_{2n}} - ...$$

Proof:

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad + \text{cyclic property of the trace}$$

Theorem 3:

$$\mathrm{Tr}\,\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n-1}} = 0$$

Proof:

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\,\gamma_5\gamma_5\gamma^\mu = -\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma_5 = -\mathrm{Tr}\,\gamma^\mu$$

Theorem 4:

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = 4i\varepsilon^{\mu\nu\rho\sigma}.$$

8 Quantum field theory via path integrals

Review of the canonical formalism for the quantisation of field theories: We are given a Lagrange density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}_0(\phi, \partial_\mu \phi) + \mathcal{L}_{\text{int}}(\phi, \partial_\mu \phi),$$

which can be split into a "free" part $\mathcal{L}_0$ (bilinear in the fields) and a part $\mathcal{L}_{\text{int}}$ describing the interactions. Each term in $\mathcal{L}_{\text{int}}$ contains at least three fields.

The momentum density conjugate to $\phi(x)$ is given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}.$$

The Hamilton function H and the Hamiltonian $\mathcal{H}$ are given by

$$H = \int d^3x \left[\pi(x) \dot{\phi}(x) - \mathcal{L} \right] = \int d^3x \, \mathcal{H}.$$

In the canonical formalism the field $\phi(x)$ and the conjugate momentum $\pi(x)$ become operators. In the Schrödinger picture the operators are time-independent. We postulate canonical (anti)-commutation relations. For bosons we require

$$\begin{aligned}
\left[\hat{\phi}_S(t_0, \vec{x}), \hat{\pi}_S(t_0, \vec{y}) \right] &= i\delta^3(\vec{x} - \vec{y}), \\
\left[\hat{\phi}_S(t_0, \vec{x}), \hat{\phi}_S(t_0, \vec{y}) \right] &= \left[\hat{\pi}_S(t_0, \vec{x}), \hat{\pi}_S(t_0, \vec{y}) \right] = 0.
\end{aligned}$$

For fermions we have (recall that the conjugate momentum is $\partial \mathcal{L} / \partial(\partial_0 \psi) = i\psi^\dagger$):

$$\begin{aligned}
\left\{ \hat{\psi}_S(t_0, \vec{x}), i\hat{\psi}_S^\dagger(t_0, \vec{y}) \right\} &= i\delta^3(\vec{x} - \vec{y}), \\
\left\{ \hat{\psi}_S(t_0, \vec{x}), \hat{\psi}_S(t_0, \vec{y}) \right\} &= \left\{ i\hat{\psi}_S^\dagger(t_0, \vec{x}), i\hat{\psi}_S^\dagger(t_0, \vec{y}) \right\} = 0.
\end{aligned}$$

To change from the Schrödinger picture to the Heisenberg picture we have the formulae

$$\begin{aligned}
\hat{\phi}_H(t, \vec{x}) &= e^{i\hat{H}(t - t_0)} \hat{\phi}_S(t_0, \vec{x}) e^{-i\hat{H}(t - t_0)}, \\
\hat{\pi}_H(t, \vec{x}) &= e^{i\hat{H}(t - t_0)} \hat{\pi}_S(t_0, \vec{x}) e^{-i\hat{H}(t - t_0)}.
\end{aligned}$$

We can also split the Hamiltonian into a "free" part and a piece describing the interactions:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}.$$

We define the field operator in the interaction picture as

$$\hat{\phi}_I(t, \vec{x}) = e^{i\hat{H}_0(t - t_0)} \hat{\phi}_S(t_0, \vec{x}) e^{-i\hat{H}_0(t - t_0)}.$$

We have the following relation between the field operators in the Heisenberg picture and the interaction picture:

$$\hat{\phi}_H(x) = \hat{U}^\dagger(t,t_0)\hat{\phi}_I(x)\hat{U}(t,t_0),$$

where

$$\hat{U}(t_1,t_0) = T\left\{\exp\left[-i\int\limits_{t_0}^{t_1} dt\, \hat{H}_I(t)\right]\right\}.$$

8.1 Trouble with the canonical quantisation of gauge bosons

The canonical formalism worked fine for the quantisation of spin 0 and spin $1/2$ particles. Let us now consider spin 1 particles. As a first example we consider the photon field without any interactions with fermions. The Lagrange density is

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The canonical momentum conjugate to A_μ is given by

$$\Pi_\mu = \frac{\partial L}{\partial \dot{A}^\mu} = -F_{0\mu}.$$

The canonical commutation relation would be given by

$$\left[\hat{A}_\mu(t_0,\vec{x}), \hat{\Pi}_\nu(t_0,\vec{y})\right] = i g_{\mu\nu}\delta^3(\vec{x}-\vec{y}).$$

If we set $\mu = \nu = 0$ we obtain

$$\left[\hat{A}_0(t_0,\vec{x}), \hat{\Pi}_0(t_0,\vec{y})\right] = i\delta^3(\vec{x}-\vec{y}).$$

On the other hand, we have

$$\Pi_0(t_0,\vec{y}) = -F_{00} = 0.$$

Thus the simple-minded application of the canonical quantisation fails. The problem is related to the invariance of the Lagrange density under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x).$$

One possibility to circumvent the problem is to eliminate the freedom of gauge transformations by putting constraints on the field A_μ. One adds a gauge-fixing term to the Lagrange density:

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

Remark: The Lagrange density is no longer gauge-invariant. The canonical momentum is now given by

$$\Pi_\mu = -F_{0\mu} - \frac{1}{\xi} g_{0\mu} (\partial^\nu A_\nu).$$

The above ad-hoc recipe works fine for abelian gauge theories like QED, but fails for non-abelian gauge theories, which are needed for the strong and weak interactions.

For a deeper understanding of the problem and its solution it is simpler to switch to a second method for the quantisation of field theories: quantisation in the path-integral formalism.

It should be noted that with an elaborate mathematical formalism it is possible to quantise non-abelian gauge fields also in the canonical formalism.

8.2 Path integrals

We start from the well-known Gaussian integral:

$$(2\pi)^{-1/2} \int\limits_{-\infty}^{\infty} dy \, \exp\left(-\frac{1}{2} a y^2\right) = a^{-1/2}.$$

This is a one-dimensional integral. We may generalise this formula to a n-dimensional integral as follows: Let A be a real symmetric positive definite matrix. Then

$$(2\pi)^{-n/2} \int\limits_{-\infty}^{\infty} dy_1 ... dy_n \, \exp\left(-\frac{1}{2} \vec{y}^T A \vec{y}\right) = (\det A)^{-1/2}.$$

We recall that the exponential of a matrix A is defined by the Taylor series

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

and that the logarithm of the matrix A is defined as a matrix such that

$$\exp(\ln A) = A.$$

We have the formula

$$\ln \det A = \mathrm{Tr} \, \ln A.$$

This formula is most easily proved by diagonalising the matrix A. Therefore we may re-write our n-fold Gaussian integral as

$$(2\pi)^{-n/2} \int\limits_{-\infty}^{\infty} dy_1 ... dy_n \, \exp\left(-\frac{1}{2} \vec{y}^T A \vec{y}\right) = \exp\left(-\frac{1}{2} \, \mathrm{Tr} \, \ln A\right).$$

We now generalise this formula in two steps: In the first step we go from a finite number of variables y_j with $j \in \{1,...,n\}$ to an infinite countable number of variables ϕ_j with $j \in \mathbb{N}$. In the second step we go from a countable number of variables ϕ_j to an over-countable number of variables $\phi(x)$ with $x \in \mathbb{R}$. In this way we arrive at a **path integral**:

$$\begin{aligned}\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int dx \int dy\, \phi(x)A(x,y)\phi(y)\right) &= [\det A\,(x,y)]^{-\frac{1}{2}} \\ &= \exp\left(-\frac{1}{2}\,\mathrm{Tr}\,\ln A\right)\end{aligned}$$

The factors of (2π) on the left-hand side have been absorbed into the path integral measure $\mathcal{D}\phi$. Let us look at a simple example:

$$\int \mathcal{D}\eta(t)\, \exp\left(-\frac{1}{2}\int\limits_{t_i}^{t_f} \eta(t)\left(-\frac{d^2}{dt^2}+\omega^2\right)\eta(t)\right) = \left[\det\left(-\frac{d^2}{dt^2}+\omega^2\right)\right]^{-\frac{1}{2}}$$

To calculate the determinant we solve the eigenvalue problem

$$\left(-\frac{d^2}{dt^2}+\omega^2\right)\psi_n(t) = \lambda_n\psi_n(t).$$

One finds

$$\lambda_n = \left(\tfrac{n\pi}{t_f-t_i}\right)^2+\omega^2,$$

and therefore

$$\det\left(-\frac{d^2}{dt^2}+\omega^2\right) = \prod_{n=1}^{\infty}\lambda_n = B\frac{\sinh\omega\left(t_f-t_i\right)}{\omega\left(t_f-t_i\right)},$$

where B is an (infinite) constant.

Note: In practice, path integrals are never calculated explicitly !

Up to now we only considered integrals, where the argument of the exponential function was quadratic in the integration variables. We consider now Gaussian integrals with a linear term. For a finite number of variables one finds

$$(2\pi)^{-n/2}\int\limits_{-\infty}^{\infty} dy_1...dy_n\, \exp\left(-\frac{1}{2}\vec{y}^TA\vec{y}+\vec{w}^T\vec{y}\right) = \exp\left(-\frac{1}{2}\,\mathrm{Tr}\,\ln A\right)\exp\left(\frac{1}{2}\vec{w}^TA^{-1}\vec{w}\right).$$

Remark: A^{-1} exists because we assumed A to be positive definite.
Generalisation to an infinite number of components:

$$\int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int dx\int dy\, \phi(x)A(x,y)\phi(y)+\int dx\, J(x)\phi(x)\right) =$$
$$\exp\left(-\frac{1}{2}\,\mathrm{Tr}\,\ln A\right)\exp\left(\frac{1}{2}\int dx\int dy\, J(x)A^{-1}(x,y)J(y)\right)$$

Let us now discuss what happens if we differentiate these formulae with respect to w_i or $J(x)$. We start with the case of a finite number of variables. Differentiation with respect to w_i at $\vec{w} = \vec{0}$ gives:

$$\begin{aligned}
&\frac{\partial}{\partial w_{i_1}} \cdots \frac{\partial}{\partial w_{i_n}} (2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \ldots dy_n \, \exp\left(-\frac{1}{2}\vec{y}^T A \vec{y} + \vec{w}^T \vec{y}\right)\Big|_{\vec{w}=0} = \\
&= (2\pi)^{-n/2} \int_{-\infty}^{\infty} dy_1 \ldots dy_n \, y_{i_1} \ldots y_{i_n} \exp\left(-\frac{1}{2}\vec{y}^T A \vec{y}\right) \\
&= \frac{\partial}{\partial w_{i_1}} \cdots \frac{\partial}{\partial w_{i_n}} \exp\left(-\frac{1}{2} \, \mathrm{Tr} \, \ln A\right) \exp\left(\frac{1}{2} \vec{w}^T A^{-1} \vec{w}\right)\Big|_{\vec{w}=0}
\end{aligned}$$

In the second line we applied the differentiation under the integral, in the third line we only substituted the result for the Gaussian integral. The main point of this formula is the equality of the second and third line: A Gaussian integral without a term $\vec{w}^T \vec{y}$ in the exponent, but with an extra factor $y_{i_1} \ldots y_{i_n}$ in front of the exponent is equal to the derivative of a Gaussian integral with a linear term with respect to the quantities w_{i_j}.

Let us now generalise this to path integrals. We first have to generalise the ordinary derivative to a **functional derivative**:

$$\frac{\delta}{\delta J(y)} Z[J(x)] = \lim_{\varepsilon \to 0} \frac{Z[J(x) + \varepsilon \delta(x-y)] - Z[J(x)]}{\varepsilon}$$

With this definition we have the following generalisation:

$$\begin{aligned}
&\frac{\partial}{\partial J(x_1)} \cdots \frac{\partial}{\partial J(x_n)} \int \mathcal{D}\phi \exp\left(-\frac{1}{2} \int dx \int dy \, \phi(x) A(x,y) \phi(y) + \int dx \, J(x) \phi(x)\right)\Big|_{J=0} = \\
&= \int \mathcal{D}\phi \, \phi(x_1) \ldots \phi(x_n) \exp\left(-\frac{1}{2} \int dx \int dy \, \phi(x) A(x,y) \phi(y)\right) \\
&= \frac{\partial}{\partial J(x_1)} \cdots \frac{\partial}{\partial J(x_n)} \exp\left(-\frac{1}{2} \, \mathrm{Tr} \, \ln A\right) \exp\left(\frac{1}{2} \int dx \int dy \, J(x) A^{-1}(x,y) J(y)\right)\Big|_{J=0} .
\end{aligned}$$

8.3 Transition amplitudes as path integrals

We start from the relation between Schrödinger and Heisenberg operators:

$$\hat{\phi}_H(t, \vec{x}) = e^{i\hat{H}(t-t_0)} \hat{\phi}_S(t_0, \vec{x}) e^{-i\hat{H}(t-t_0)}$$

Here, t_0 is a reference time, where $\hat{\phi}_H(t_0, \vec{x}) = \hat{\phi}_S(t_0, \vec{x})$. Let $|\phi(t_j, \vec{x}), t\rangle_S$ be a Schrödinger state, which for $t = t_j$ is an eigenstate of the Schrödinger field operator $\hat{\phi}_S(t_0, \vec{x})$ with eigenvalue $\phi(t_j, \vec{x})$:

$$\hat{\phi}_S(t_0, \vec{x}) |\phi(t_j, \vec{x}), t_j\rangle_S = \phi(t_j, \vec{x}) |\phi(t_j, \vec{x}), t_j\rangle_S$$

Denote the corresponding Heisenberg state by

$$|\phi(t_j,\vec{x})\rangle_H = e^{i\hat{H}(t-t_0)}|\phi(t_j,\vec{x}),t\rangle_S.$$

$|\phi(t_j,\vec{x})\rangle_H$ is an eigenstate of $\hat{\phi}_H(t_j,\vec{x})$ with eigenvalue $\phi(t_j,\vec{x})$:

$$\hat{\phi}_H(t_j,\vec{x})|\phi(t_j,\vec{x})\rangle_H = \phi(t_j,\vec{x})|\phi(t_j,\vec{x})\rangle_H.$$

Proof:

$$\begin{aligned}\hat{\phi}_H(t_j,\vec{x})|\phi(t_j,\vec{x})\rangle_H &= \left(e^{i\hat{H}(t_j-t_0)}\hat{\phi}_S(t_0,\vec{x})e^{-i\hat{H}(t_j-t_0)}\right)\left(e^{i\hat{H}(t_j-t_0)}|\phi(t_j,\vec{x}),t_j\rangle_S\right)\\ &= e^{i\hat{H}(t_j-t_0)}\hat{\phi}_S(t_0,\vec{x})|\phi(t_j,\vec{x}),t_j\rangle_S = e^{i\hat{H}(t_j-t_0)}\phi(t_j,\vec{x})|\phi(t_j,\vec{x}),t_j\rangle_S\\ &= \phi(t_j,\vec{x})|\phi(t_j,\vec{x})\rangle_H\end{aligned}$$

As a short-hand notation we write

$$\begin{aligned}|\phi_k,t_j\rangle &= |\phi_k(t_j,\vec{x})\rangle_H,\\ |\phi_k\rangle &= |\phi_k(t_j,\vec{x}),t_j\rangle_S.\end{aligned}$$

We are interested in the transition amplitude

$$\langle\phi_f,t_f|\phi_i,t_i\rangle.$$

$|\phi_i,t_i\rangle$ is a state with eigenvalue $\phi_i(t_i,\vec{x})$ at $t=t_i$ and similar for $|\phi_f,t_f\rangle$. We divide the time interval (t_f-t_i) into $n+1$ small sub-intervals with time steps at

$$t_i,t_1,t_2,...,t_n,t_f.$$

At each intermediate time step we insert a complete set of states

$$\int \mathcal{D}\phi_j(\vec{x})\ |\phi_j,t_j\rangle\langle\phi_j,t_j| = 1.$$

Therefore

$$\langle\phi_f,t_f|\phi_i,t_i\rangle = \int\mathcal{D}\phi_n(\vec{x})...\int\mathcal{D}\phi_1(\vec{x})\ \langle\phi_f,t_f|\phi_n,t_n\rangle\langle\phi_n,t_n|\phi_{n-1},t_{n-1}\rangle...\langle\phi_1,t_1|\phi_i,t_i\rangle.$$

Let us study $\langle\phi_{j+1},t_{j+1}|\phi_j,t_j\rangle$. If the time interval $(t_{j+1}-t_j)$ is small, we have

$$\begin{aligned}\langle\phi_{j+1},t_{j+1}|\phi_j,t_j\rangle &= \left(\langle\phi_{j+1}|e^{-i\hat{H}(t_{j+1}-t_0)}\right)\left(e^{i\hat{H}(t_j-t_0)}|\phi_j,t_j\rangle\right) = \langle\phi_{j+1}|e^{-i\hat{H}(t_{j+1}-t_j)}|\phi_j\rangle\\ &\approx \langle\phi_{j+1}|1-i(t_{j+1}-t_j)\hat{H}|\phi_j\rangle\end{aligned}$$

Let us first consider a simple case where $\hat{H}$ is replaced by a function $f(\hat{\phi})$, which depends only on $\hat{\phi}$, but not on $\hat{\pi}$. Then

$$\langle\phi_{j+1}|f(\hat{\phi})|\phi_j\rangle = f(\phi_j)\langle\phi_{j+1}|\phi_j\rangle = f(\phi_j)\delta(\phi_{j+1}-\phi_j)$$

We rewrite this as

$$\langle\phi_{j+1}|f(\hat\phi)|\phi_j\rangle \;=\; f(\phi_j)\int \mathcal{D}\pi_j(\vec x)\;\; \exp\left[i\int d^3x\,\pi_j(\phi_{j+1}-\phi_j)\right]$$

Note that factors of 2π are absorbed into the integral measure. Next we consider the case where the Hamiltonian is replaced by a function $g(\hat\pi)$, which depends only on $\hat\pi$. We introduce a complete set of momentum eigenstates and obtain

$$\langle\phi_{j+1}|g(\hat\pi)|\phi_j\rangle \;=\; \int \mathcal{D}\pi_j(\vec x)\;\; g(\pi_j)\exp\left[i\int d^3x\,\pi_j(\phi_{j+1}-\phi_j)\right]$$

Thus if $\hat H$ contains only terms of the form $f(\hat\phi)$ and $g(\hat\pi)$, its matrix element can be written as

$$\langle\phi_{j+1}|\hat H(\hat\phi,\hat\pi)|\phi_j\rangle \;=\; \int \mathcal{D}\pi_j(\vec x)\;\; H(\phi_j,\pi_j)\exp\left[i\int d^3x\,\pi_j(\phi_{j+1}-\phi_j)\right]$$

In general this formula will not hold for arbitrary $\hat H$, since the order of a product $\hat\phi\hat\pi$ matters on the left side (where $\hat\phi$ and $\hat\pi$ appear as operators), but not on the right side.

If this formula holds, the Hamiltonian is said to be in **Weyl order**. Any Hamiltonian can be put into a Weyl order by commuting $\hat\phi$'s and $\hat\pi$'s.

Assuming now Weyl order, we find

$$\begin{aligned}
\langle\phi_{j+1}|1-i(t_{j+1}-t_j)\hat H|\phi_j\rangle &= \int \mathcal{D}\pi_j(\vec x)\;\; \big(1-i(t_{j+1}-t_j)H(\phi_j,\pi_j)\big)\exp\left[i\int d^3x\,\pi_j(\phi_{j+1}-\phi_j)\right] \\
&\approx \int \mathcal{D}\pi_j(\vec x)\;\; \exp\left[i\left(\int d^3x\,\pi_j(\phi_{j+1}-\phi_j)\right)-i(t_{j+1}-t_j)H(\phi_j,\pi_j)\right] \\
&= \int \mathcal{D}\pi_j(\vec x)\;\; \exp\left[i(t_{j+1}-t_j)\int d^3x\;\big(\pi_j\dot\phi_j-\mathcal{H}(\phi_j,\pi_j)\big)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle\phi_f,t_f|\phi_i,t_i\rangle &= \int \mathcal{D}\phi_n(\vec x)\ldots\int \mathcal{D}\phi_1(\vec x)\;\; \langle\phi_f,t_f|\phi_n,t_n\rangle\langle\phi_n,t_n|\phi_{n-1},t_{n-1}\rangle\ldots\langle\phi_1,t_1|\phi_i,t_i\rangle \\
&= \int \mathcal{D}\phi(t,\vec x)\int \mathcal{D}\pi(t,\vec x)\;\; \exp\left[i\int dt\int d^3x\,\pi\dot\phi-\mathcal{H}(\phi,\pi)\right].
\end{aligned}$$

In most cases $\mathcal{H}(\phi,\pi)$ will be quadratic in π. We can then complete the square, perform a Wick rotation and integrate over $\mathcal{D}\pi(t,\vec x)$. For example, for

$$\mathcal{H} \;=\; \frac{1}{2}\pi^2+\frac{1}{2}\left(\vec\nabla\phi\right)^2+V(\phi)\,, \qquad \mathcal{L} \;=\; \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)-V(\phi)$$

we have

$$
\begin{aligned}
& \int \mathcal{D}\pi(t,\vec{x})\ \exp\left[i\int d^4x \pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - V(\phi)\right] \\
&= \int \mathcal{D}\pi(t,\vec{x})\ \exp\left[i\int d^4x \left(-\frac{1}{2}\left(\pi-\dot{\phi}\right)^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - V(\phi)\right)\right] \\
&= \mathcal{N} \exp\left[i\int d^4x \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - V(\phi)\right] \\
&= \mathcal{N} \exp\left[i\int d^4x \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - V(\phi)\right].
\end{aligned}
$$

One obtains

$$
\begin{aligned}
\langle \phi_f, t_f | \phi_i, t_i \rangle &= \int \mathcal{D}\phi(t,\vec{x}) \int \mathcal{D}\pi(t,\vec{x})\ \exp\left[i\int dt \int d^3x\, \pi\dot{\phi} - \mathcal{H}(\phi,\pi)\right] \\
&= \int \mathcal{D}\phi(x)\ \exp\left[i\int d^4x\, \mathcal{L}\right].
\end{aligned}
$$

8.4 Correlation functions

In the previous section we found

$$
\langle \phi_f, t_f | \phi_i, t_i \rangle = \int \mathcal{D}\phi(x)\ \exp\left[i\int d^4x\, \mathcal{L}\right].
$$

The time interval goes from t_i to t_f, in all other respects this formula is manifestly Lorentz invariant. Any other symmetries that the Lagrangian may have are preserved by the functional integral as long as the path integral measure is also invariant under these symmetries. This will be important for internal symmetries related to gauge groups.

We now would like to make the next step and define quantum field theory through path integrals. We have to find a functional formula to compute correlation functions like

$$
\langle \Omega | T\hat{\phi}(x_1)\hat{\phi}(x_2) | \Omega \rangle,
$$

and to show its equivalence with the canonical operator formalism. Let us consider

$$
\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2) \exp\left[i\int\limits_{-T}^{T} dt \int d^3x\, \mathcal{L}(\phi)\right],
$$

where the boundary conditions on the path integral are

$$
\phi(-T,\vec{x}) = \phi_a(\vec{x}), \qquad \phi(T,\vec{x}) = \phi_b(\vec{x}).
$$

We break up the functional integral as follows:

$$\int \mathcal{D}\phi(x) \;=\; \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int\limits_{\phi(x_1^0,\vec{x})=\phi_1(\vec{x}),\phi(x_2^0,\vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x)$$

The main functional integral $\int \mathcal{D}\phi(x)$ is now constrained at times x_1^0 and x_2^0 in addition to the endpoints $-T$ and T. With this decomposition the extra factors $\phi(x_1)$ and $\phi(x_2)$ in the original path integral become $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$. If $x_1^0 < x_2^0$:

$$\begin{aligned}
&\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2)\exp\left[i\int_{-T}^{T} dt \int d^3x\, \mathcal{L}(\phi)\right] = \\
&= \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int\limits_{\phi(x_1^0,\vec{x})=\phi_1(\vec{x}),\phi(x_2^0,\vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x)\; \phi(x_1)\phi(x_2)\exp\left[i\int_{-T}^{T} dt \int d^3x\, \mathcal{L}(\phi)\right] \\
&= \int \mathcal{D}\phi_1(\vec{x})\; \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\; \phi_2(\vec{x}) \int\limits_{\phi(x_1^0,\vec{x})=\phi_1(\vec{x}),\phi(x_2^0,\vec{x})=\phi_2(\vec{x})} \mathcal{D}\phi(x)\; \exp\left[i\int_{-T}^{T} dt \int d^3x\, \mathcal{L}(\phi)\right] \\
&= \int \mathcal{D}\phi_1(\vec{x})\; \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\; \phi_2(\vec{x})\; \langle\phi_b,T|\phi_2,x_2^0\rangle\langle\phi_2,x_2^0|\phi_1,x_1^0\rangle\langle\phi_1,x_1^0|\phi_a,-T\rangle.
\end{aligned}$$

Since

$$\hat{\phi}_H(t,\vec{x})|\phi(\vec{x}),t\rangle \;=\; \phi(\vec{x})|\phi(\vec{x}),t\rangle,$$

we can turn the fields $\phi_i(\vec{x})$ into Heisenberg operators $\hat{\phi}_i(\vec{x})$. Using in addition the completeness relation

$$\int \mathcal{D}\phi(\vec{x})\; |\phi,t\rangle\langle\phi,t| \;=\; 1,$$

we obtain

$$\begin{aligned}
&\int \mathcal{D}\phi_1(\vec{x})\; \phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\; \phi_2(\vec{x})\; \langle\phi_b,T|\phi_2,x_2^0\rangle\langle\phi_2,x_2^0|\phi_1,x_1^0\rangle\langle\phi_1,x_1^0|\phi_a,-T\rangle = \\
&\quad = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\; \langle\phi_b,T|\hat{\phi}(x_2)|\phi_2,x_2^0\rangle\langle\phi_2,x_2^0|\hat{\phi}(x_1)|\phi_1,x_1^0\rangle\langle\phi_1,x_1^0|\phi_a,-T\rangle \\
&\quad = \left\langle\phi_b,T\left|\hat{\phi}(x_2)\hat{\phi}(x_1)\right|\phi_a,-T\right\rangle.
\end{aligned}$$

If we had the order $x_1^0 > x_2^0$ we would have found

$$\left\langle\phi_b,T\left|\hat{\phi}(x_1)\hat{\phi}(x_2)\right|\phi_a,-T\right\rangle.$$

In summary we have shown that

$$\int\limits_{\substack{\phi(-T,\vec{x})=\phi_a(\vec{x}),\\ \phi(T,\vec{x})=\phi_b(\vec{x})}} \mathcal{D}\phi(x)\; \phi(x_1)\phi(x_2)\exp\left[i\int_{-T}^{T} dt \int d^3x\, \mathcal{L}(\phi)\right] \;=\; \left\langle\phi_b,T\left|T\hat{\phi}(x_1)\hat{\phi}(x_2)\right|\phi_a,-T\right\rangle.$$

If we replace the Heisenberg states by Schrödinger states

$$|\phi(\vec{x}),t\rangle \;=\; e^{i\hat{H}t}|\phi(t,\vec{x}),t\rangle_S \;=\; e^{i\hat{H}t}|\phi(\vec{x})\rangle,$$

we have

$$\left\langle \phi_b \left| e^{-i\hat{H}T}\; T\hat{\phi}(x_1)\hat{\phi}(x_2)\; e^{i\hat{H}(-T)} \right| \phi_a \right\rangle.$$

As in the canonical operator formalism we can now send $T \to \infty(1-i\varepsilon)$ to project out the ground state

$$e^{-i\hat{H}T}\,|\phi_a\rangle \;=\; \sum_n e^{-iE_nT}|n\rangle\langle n|\phi_a\rangle \to \langle\Omega|\phi_a\rangle \lim_{T\to\infty(1-i\varepsilon)} e^{-iE_0T}|\Omega\rangle.$$

The phase and the overlap factor drop out if we divide by the same quantity without the field insertions and we obtain the final formula

$$\left\langle \Omega \left| T\hat{\phi}(x_1)\hat{\phi}(x_2) \right| \Omega \right\rangle \;=\; \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\phi(x)\; \phi(x_1)\phi(x_2) \exp\left[i\int d^4x\, \mathcal{L}(\phi)\right]}{\int \mathcal{D}\phi(x)\; \exp\left[i\int d^4x\, \mathcal{L}(\phi)\right]}.$$

This expresses the two-point correlation function in terms of path integrals. For higher correlation functions one obtains

$$\left\langle \Omega \left| T\hat{\phi}(x_1)...\hat{\phi}(x_n) \right| \Omega \right\rangle \;=\; \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\phi(x)\; \phi(x_1)...\phi(x_n) \exp\left[i\int d^4x\, \mathcal{L}(\phi)\right]}{\int \mathcal{D}\phi(x)\; \exp\left[i\int d^4x\, \mathcal{L}(\phi)\right]}.$$

Let us now introduce the "generating functional"

$$Z[J(x)] \;=\; \mathcal{N} \int \mathcal{D}\phi(x)\; \exp\left[i \int d^4x\, \mathcal{L}(\phi) + J(x)\phi(x)\right].$$

We have

$$\left\langle \Omega \left| T\hat{\phi}(x_1)...\hat{\phi}(x_n) \right| \Omega \right\rangle \;=\; \left.\frac{(-i)^n}{Z[0]} \frac{\delta^n Z[J(x)]}{\delta J(x_1)...\delta J(x_n)}\right|_{J=0}.$$

The functional $Z[J]$ generates all Green functions:

$$Z[J] \;=\; Z[0] \sum_n \frac{i^n}{n!} \int d^4x_1 ... d^4x_n\; \left\langle \Omega \left| T\hat{\phi}(x_1)...\hat{\phi}(x_n) \right| \Omega \right\rangle\; J(x_1)...J(x_n).$$

8.5 Fermions in the path integral formalism

8.5.1 Grassmann numbers

Ordinary number commute:

$$\left[x_i, x_j\right] \;=\; 0.$$

The Grassmann algebra consists of anti-commuting numbers

$$\left\{\theta_i, \theta_j\right\} = 0.$$

The differentiation is defined by

$$\frac{d}{d\theta} 1 = 0, \qquad \frac{d}{d\theta}\theta = 1.$$

If we have several Grassmann variables, we first bring the Grassmann variable we want to differentiate immediately after the differentiation operator. Thus we have

$$\frac{\partial}{\partial \theta_j}\left(\theta_1 ... \theta_j ... \theta_m\right) = (-1)^{j-1} \theta_1 ... \hat{\theta}_j ... \theta_m,$$

where the hat indicates that the corresponding variable has to be omitted. Note that

$$\frac{\partial}{\partial \theta_i}\frac{\partial}{\partial \theta_j} F = -\frac{\partial}{\partial \theta_j}\frac{\partial}{\partial \theta_i} F.$$

The Taylor expansion of a function $F(\theta)$ depending on a Grassmann variable θ is given by

$$F(\theta) = F_0 + F_1 \theta.$$

The differential $d\theta$ is also a Grassmann variable:

$$\{\theta, d\theta\} = 0.$$

Integration over a Grassmann variable is defined by

$$\int d\theta = 0, \qquad \int d\theta\theta = 1.$$

Multiple integrals are defined by iteration:

$$\int d\theta_1 d\theta_2 F(\theta_1, \theta_2) = \int d\theta_1 \left(\int d\theta_2 F(\theta_1, \theta_2) \right).$$

Let us now consider

$$\begin{aligned}
&\int d\theta_1 d\theta_2 ... d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 ... d\bar{\theta}_n \exp\left(\bar{\theta}_i A_{ij} \theta_j\right) = \\
&= \int d\theta_1 d\theta_2 ... d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 ... d\bar{\theta}_n \left[1 + \bar{\theta}_i A_{ij} \theta_j + ... + \frac{1}{n!}\left(\bar{\theta}_i A_{ij} \theta_j\right)^n\right] \\
&= \int d\theta_1 d\theta_2 ... d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 ... d\bar{\theta}_n \frac{1}{n!}\left(\bar{\theta}_i A_{ij} \theta_j\right)^n \\
&= \int d\theta_1 d\theta_2 ... d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 ... d\bar{\theta}_n \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \sum_{i_1,...,i_n,j_1,...,j_n} A_{i_1 j_1} ... A_{i_n j_n} \bar{\theta}_{i_1} ... \bar{\theta}_{i_n} \theta_{j_1} ... \theta_{j_n}
\end{aligned}$$

$$= \int d\theta_1 d\theta_2 ... d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 ... d\bar{\theta}_n \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \sum_{i_1,...,i_n,j_1,...,j_n} \varepsilon_{i_1...i_n} \varepsilon_{j_1...j_n} A_{i_1 j_1} ... A_{i_n j_n} \bar{\theta}_1 ... \bar{\theta}_n \theta_1 ... \theta_n$$

$$= \int d\theta_1 d\theta_2 ... d\theta_n d\bar{\theta}_1 d\bar{\theta}_2 ... d\bar{\theta}_n (-1)^{\frac{n(n-1)}{2}} \sum_{j_1,...,j_n} \varepsilon_{j_1...j_n} A_{1 j_1} ... A_{n j_n} \bar{\theta}_1 ... \bar{\theta}_n \theta_1 ... \theta_n$$

$$= (-1)^{\frac{n(n-1)}{2}} \sum_{j_1,...,j_n} \varepsilon_{j_1...j_n} A_{1 j_1} ... A_{n j_n}$$

$$= (-1)^{\frac{n(n-1)}{2}} \det A.$$

The limit $n \to \infty$ yields a path integral over a Grassmann field. We thus arrive at the important formula

$$\det A \quad \sim \quad \int \mathcal{D}\psi(x)\ \mathcal{D}\bar{\psi}(x)\ \exp \int d^4x\, d^4y\ \bar{\psi}(x) A(x,y) \psi(y).$$

8.5.2 Path integrals with fermions

For fermions we considered up to now the Lagrange density of free fermions and the Lagrange density of interacting fermions in the Yukawa model:

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Klein-Gordon}} - g\bar{\psi}\psi\phi, \\
\mathcal{L}_{\text{Dirac}} &= i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi, \\
\mathcal{L}_{\text{Klein-Gordon}} &= \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2.
\end{aligned}$$

With the help of Grassmann numbers we may write down the generating functional

$$Z[J(x), \eta(x), \bar{\eta}(x)] = \mathcal{N} \int \mathcal{D}\phi(x) \mathcal{D}\psi(x) \mathcal{D}\bar{\psi}(x)\ \exp \left[i \int d^4x\ \mathcal{L}(\phi) + J(x)\phi(x) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) \right]$$

Here $\bar{\psi}(x)$, $\psi(x)$, $\bar{\eta}(x)$ and $\eta(x)$ are fields of Grassmann nature. As before we obtain the Green functions by differentiation. For example:

$$\left\langle \Omega \left| T \hat{\psi}_\alpha(x_1) \hat{\bar{\psi}}_\beta(x_2) \right| \Omega \right\rangle = \left. \frac{(-i)^2}{Z[0,0,0]} \frac{\delta^2 Z[J(x), \eta(x), \bar{\eta}(x)]}{\delta \bar{\eta}_\alpha(x_1) \delta(-\eta_\beta(x_2))} \right|_{J=0}$$

The additional minus sign in the differentiation with respect to $\eta(x_2)$ comes from anti-commuting

$$\bar{\psi}(x)\eta(x) = -\eta(x)\bar{\psi}(x).$$

8.6 The reduction formula of Lehmann, Symanzik and Zimmermann

In the canonical operator formalism we stated the formula that the non-trivial part of the S-matrix can be computed as follows

$$\left\langle \vec{p}_1 \vec{p}_2 ... \left| i\hat{T} \right| \vec{p}_A \vec{p}_B \right\rangle = \lim_{T \to \infty(1-i\varepsilon)} \left({}_0\langle \vec{p}_1 \vec{p}_2 ... | T \exp \left[-i \int\limits_{-T}^{T} dt H_I(t) \right] | \vec{p}_A \vec{p}_B \rangle_0 \right)_{\text{connected,amputated}}.$$

We now derive in the path integral formalism the reduction formula of Lehmann, Symanzik and Zimmermann, which explains why propagators of external legs are amputated. In the path integrals formalism we already showed that

$$\left\langle \Omega \left| T\hat{\phi}(x_1)...\hat{\phi}(x_n) \right| \Omega \right\rangle = \frac{\int \mathcal{D}\phi(x)\, \phi(x_1)...\phi(x_n) \exp\left[i \int d^4x\, \mathcal{L}(\phi) \right]}{\int \mathcal{D}\phi(x)\, \exp\left[i \int d^4x\, \mathcal{L}(\phi) \right]},$$

where the boundary conditions on the path integral are

$$\lim_{T\to\infty} \phi(-T,\vec{x}) = \lim_{T\to\infty} \phi(T,\vec{x}) = 0.$$

For the computation of scattering amplitudes we would like to have as boundary condition not the vacuum but an n particle state. If we assume that interaction are only relevant within a finite volume, we can take this n particle state as the superposition of n non-interacting one-particle states. We call such a state an asymptotic state. Asymptotic fields are solutions of the non-interacting theory, e.g free fields. The general solution for the free field theory is given as a Fourier expansion:

$$\phi_{\text{asymp}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} \left(a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \right)$$

Note that $a(k)$ and $a^\dagger(k)$ are here c-numbers, not operators. With boundary conditions at the remote past/future:

$$t \to +\infty: \qquad \phi_{\text{asymp}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} a(k) e^{-ikx},$$
$$t \to -\infty: \qquad \phi_{\text{asymp}}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} a^\dagger(k) e^{ikx}.$$

If we give k_0 a small imaginary part $k_0 \to k_0 - i\varepsilon$, we can use the general formula for both cases. If we consider a scalar field theory, the asymptotic field satisfies the Klein-Gordon equation

$$\left(\Box + m^2 \right) \phi_{\text{asymp}}(x) = 0.$$

Consider now:

$$Z_{\text{asymp}}[J] = \int\limits_{\lim \phi = \phi_{\text{asymp}}} \mathcal{D}\phi\, \exp\left[i \int d^4x\, \mathcal{L}(\phi) + J(x)\phi(x) \right]$$

With

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_{\text{int}}(\phi),$$
$$\exp\left[i \int d^4x\, \mathcal{L}_{\text{int}}(\phi) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \int d^4x\, \mathcal{L}_{\text{int}}(\phi) \right)^n,$$
$$i\phi(x) \exp\left[i \int d^4y\, \mathcal{L}_0(\phi) + J(y)\phi(y) \right] = \frac{\delta}{\delta J(x)} \exp\left[i \int d^4y\, \mathcal{L}_0(\phi) + J(y)\phi(y) \right],$$

one arrives at

$$Z_{\text{asymp}}[J] = \exp\left[i\int d^4x\mathcal{L}_{\text{int}}\left(-i\frac{\delta}{\delta J(x)}\right)\right]\int\limits_{\lim\phi=\phi_{\text{asymp}}}\mathcal{D}\phi\ \exp\left[i\int d^4y\ \mathcal{L}_0(\phi)+J(y)\phi(y)\right].$$

Let us now define the free-field functional $Z_{\text{asymp},0}[J]$

$$Z_{\text{asymp},0}[J] = \int\limits_{\lim\tilde{\phi}=\phi_{\text{asymp}}}\mathcal{D}\tilde{\phi}\ \exp\left[i\int d^4x\ \mathcal{L}_0(\tilde{\phi})+J(x)\tilde{\phi}(x)\right]$$

and change the integration variables accoridng to

$$\tilde{\phi}(x) = \phi(x)+\phi_{\text{asymp}}(x).$$

Then

$$Z_{\text{asymp},0}[J] = \int\limits_{\lim\phi=0}\mathcal{D}\phi\ \exp\left[i\int d^4x\ \mathcal{L}_0(\phi+\phi_{\text{asymp}})+J(x)\phi(x)+J(x)\phi_{\text{asymp}}(x)\right].$$

Note that now the boundary conditions are

$$\lim_{t\to\pm\infty}\phi = 0.$$

Consider now the example of a scalar field

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\frac{1}{2}m^2\phi^2.$$

Then

$$\begin{aligned} i\int d^4x\ \mathcal{L}_0\left(\phi+\phi_{\text{asymp}}\right) &= i\int d^4x\Big\{\mathcal{L}_0(\phi)-\frac{1}{2}\phi_{\text{asymp}}\left(\Box+m^2\right)\phi_{\text{asymp}} \\ &\quad -\frac{1}{2}\phi\left(\Box+m^2\right)\phi_{\text{asymp}}-\frac{1}{2}\left[\left(\Box+m^2\right)\phi_{\text{asymp}}\right]\phi\Big\} \\ &= i\int d^4x\ \mathcal{L}_0(\phi), \end{aligned}$$

since ϕ_{asymp} satisfies the Klein-Gordon equation:

$$\left(\Box+m^2\right)\phi_{\text{asymp}} = 0.$$

In this case we have

$$Z_{\text{asymp},0}[J] = \int\limits_{\lim\phi=0}\mathcal{D}\phi\ \exp\left[i\int d^4y\ J(y)\phi_{\text{asymp}}(y)\right]\exp\left[i\int d^4x\ \mathcal{L}_0(\phi)+J(x)\phi(x)\right].$$

We now write

$$\int\limits_{\lim\phi=0} \mathcal{D}\phi\ \exp\left[i\int d^4x\ \mathcal{L}_0(\phi)+J(x)\phi(x)\right] \ = \ \exp\left[-\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y,z)J(z)\right].$$

For the example discussed above (scalar field theory) we have

$$\Delta(x-y) \ = \ \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{1}{p^2-m^2+i\varepsilon}.$$

Since

$$\left(\Box_x+m^2\right)\Delta(x-y) \ = \ -\delta(x-y)$$

and

$$\begin{aligned}&\frac{\delta}{\delta J(x)}\exp\left[-\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y,z)J(z)\right] = \\ &\quad -i\int d^4w\,\Delta(x-w)J(w)\exp\left[-\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y,z)J(z)\right],\end{aligned}$$

we have

$$\left(\Box_x+m^2\right)\frac{\delta}{\delta J(x)}\exp\left[-\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y,z)J(z)\right] \ = \ iJ(x)\exp\left[-\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y,z)J(z)\right].$$

Therefore

$$\begin{aligned} Z_{\text{asymp}}[J] \ &= \ \exp\left[\int d^4x\ \phi_{\text{asymp}}(x)\cdot\left(\Box_x+m^2\right)\frac{\delta}{\delta J(x)}\right] \\ &\quad \times\exp\left[i\int d^4y\mathcal{L}_{\text{int}}\left(-i\frac{\delta}{\delta J(y)}\right)\right] \\ &\quad \times\int\limits_{\lim\phi=0}\mathcal{D}\phi\ \exp\left[i\int d^4z\left(\mathcal{L}_0(\phi)+J(z)\phi(z)\right)\right].\end{aligned}$$

Let us now define

$$Z[J] \ = \ \int\limits_{\lim\phi=0}\mathcal{D}\phi\ \exp\left[i\int d^4z\ \mathcal{L}(\phi)+J(z)\phi(z)\right].$$

Define the Green functions as functional derivatives of $Z[J]$:

$$\begin{aligned} G^n(x_1,\ldots,x_n) \ &= \ \left\langle 0\middle|T\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\middle|0\right\rangle \\ &= \ \frac{(-i)^n}{Z[0]}\left.\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}\end{aligned}$$

Then

$$Z_{\text{asymp}}[0] = \sum \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \phi_{\text{asymp}}(x_1) \dots \phi_{\text{asymp}}(x_n) \left(\Box_{x_1} + m^2\right) \dots \left(\Box_{x_n} + m^2\right) G^n(x_1, \dots, x_n).$$

Define now the Fourier transform of the Green functions by

$$G^n(x_1, \dots, x_n) = \int \frac{d^4p_1}{(2\pi)^4} \dots \frac{d^4p_n}{(2\pi)^4} e^{-i\sum p_j x_j} (2\pi)^4 \delta^4(p_1 + \dots + p_n) \tilde{G}^n(p_1, \dots, p_n)$$

and the truncated (amputated) Green function in momentum space by

$$\tilde{G}^n_{\text{trunc}}(p_1, \dots, p_n) = \left(\frac{i}{p_1^2 - m^2}\right)^{-1} \dots \left(\frac{i}{p_n^2 - m^2}\right)^{-1} \tilde{G}^n(p_1, \dots, p_n).$$

Then

$$\begin{aligned} Z_{\text{asymp}}[0] = & \sum \frac{1}{n!} \int \frac{d^4p_j}{(2\pi)^4} (2\pi)^4 \delta^4\left(\sum p_k\right) \tilde{G}^n_{\text{trunc}}(p_1, \dots, p_n) \\ & \int d^4x_1 \dots d^4x_n e^{-i\sum p_j x_j} \phi_{\text{asymp}}(x_1) \dots \phi_{\text{asymp}}(x_n). \end{aligned}$$

Consider

$$\begin{aligned} \int d^4x\, e^{-ipx} \phi_{\text{asymp}}(x) &= \int d^4x\, e^{-ipx} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} \left[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right] \\ &= \int d^3k \frac{2\pi}{\sqrt{2k_0}} \left[a(k)\delta^4(p+k) + a^\dagger(k)\delta^4(p-k)\right] \end{aligned}$$

For $k_0 > 0$ and $p_0 > 0$ only the second term contributes. For $k_0 > 0$ and $p_0 < 0$ only the first term contributes. One obtains

$$Z_{\text{asymp}}[0] = \sum \frac{1}{n!} \int \frac{d^3p_j}{(2\pi)^3} \frac{1}{\sqrt{|2p_0|}} (2\pi)^4 \delta^4\left(\sum p_k\right) \tilde{G}^n_{\text{trunc}}(p_1, \dots, p_n) \prod a(-p_i) \prod a^\dagger(p_j).$$

9 Gauge theories

We have seen that electrodynamics can be described by a gauge potential $A_\mu(x)$.

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This Lagrange density is invariant under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda(x).$$

We can write this gauge transformation also as

$$A_\mu(x) \rightarrow U(x) \left(A_\mu(x) + i\partial_\mu \right) U^\dagger(x),$$

with

$$U(x) = e^{-i\Lambda(x)}.$$

The gauge symmetry is given by a $U(1)$ group: This is an abelian group, whose elements can be parameterised with a coordinate φ as follows:

$$e^{-i\varphi}, \qquad 0 \leq \varphi < 2\pi.$$

This is obviously a group:

$$\begin{aligned} e^{-i\varphi_1} \cdot e^{-i\varphi_2} &= e^{-i(\varphi_1+\varphi_2)}, \\ \left(e^{-i\varphi}\right)^{-1} &= e^{i\varphi}. \end{aligned}$$

It is also a one-dimensional compact manifold (e.g. the circle line).

9.1 Lie groups und Lie algebras

A Lie group is a group G which is also an analytic manifold such that the mapping $(a,b) \rightarrow ab^{-1}$ of the product manifold $G \times G$ into G is analytic.
A Lie algebra over a commutative ring K is a K-module A together with a mapping $x \otimes y \rightarrow [x,y]$ such that for $x,y,z \in A$:

$$\begin{aligned} [x,x] &= 0, \\ [x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0. \end{aligned}$$

Elements of a Lie group are written in terms of the generators as

$$g = \exp\left(-iT^a \alpha_a\right).$$

The generators T^a satisfy a Lie algebra, e.g. the commutators of generators are linear combinations of the generators, i.e.

$$[T^a, T^b] = i f^{abc} T^c.$$

Note that in the mathematical literature the convention for the definition of the generators is usually such that no explicit factors of i appear in the formulae above. The convention used in the physical literature (which is adopted here) ensures that the generators for the unitary groups are hermitian matrices.

Examples of Lie groups:

- $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$: group of non-singular $n \times n$ matrices with n^2 real parameters ($GL(n,\mathbb{R})$), respectively $2n^2$ real parameters ($GL(n,\mathbb{C})$).
- $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$: $\det A = 1$; $SL(n,\mathbb{R})$ has $n^2 - 1$ real parameters; $SL(n,\mathbb{C})$ has $2(n^2-1)$ real parameters.
- $O(n)$: $RR^T = 1$.
- $SO(n)$: $RR^T = 1$ and $\det R = 1$.
- $U(n)$: $UU^\dagger = 1$; n^2 real parameters.
- $SU(n)$: $UU^\dagger = 1$ and $\det U = 1$; $n^2 - 1$ real parameters.
- $Sp(n)$: Invariance group of

$$\sum_{j=1}^{n} \left(x_j y_{j+n} - x_{j+n} y_j \right).$$

A Lie algebra is simple if it is non-Abelian and has no non-trivial ideals.

A Lie algebra is called semi-simple if it has no non-trivial Abelian ideals.
A simple Lie algebra is also semi-simple.
Semi-simple groups are a direct product of simple groups. The compact simple Lie algebras are

$$\begin{aligned} A_n &= SU(n+1), \\ B_n &= SO(2n+1), \\ C_n &= Sp(n), \\ D_n &= SO(2n). \end{aligned}$$

The exceptional groups are

$$E_6, E_7, E_8, F_4, G_2.$$

9.2 Special unitary Lie groups

We discuss here $SU(N)$. The standard normalisation is

$$\mathrm{Tr}\left(T^a T^b\right) = T_R \delta^{ab} = \frac{1}{2}\delta^{ab}$$

As an example we consider first the group $SU(2)$. This group has three generators I^1, I^2 und I^3, which are proportional to the Pauli matrices:

$$I^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As a further example we consider $SU(3)$. Here we have eight generators λ^a, $a = 1, ..., 8$, which are called Gell-Mann matrices.

$$\lambda^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The Fierz identity reads for $SU(N)$:

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right).$$

Proof: T^a and the unit matrix form a basis of the $N \times N$ hermitian matrices, therefore any hermitian matrix A can be written as

$$A = c_0 1 + c_a T^a.$$

The constants c_0 and c_a are determined using the normalisation condition and the fact that the T^a are traceless:

$$\begin{aligned} c_0 &= \frac{1}{N}\mathrm{Tr}(A), \\ c_a &= 2\mathrm{Tr}(T^a A). \end{aligned}$$

Therefore

$$A_{lk}\left(2T^a_{ij}T^a_{kl} + \frac{1}{N}\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}\right) = 0.$$

This has to hold for an arbitrary A, therefore the Fierz identity follows. Useful formulae involving traces:

$$\begin{aligned} \mathrm{Tr}\,(T^a X)\,\mathrm{Tr}\,(T^a Y) &= \frac{1}{2}\left[\mathrm{Tr}\,(XY) - \frac{1}{N}\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(Y)\right], \\ \mathrm{Tr}\,(T^a X T^a Y) &= \frac{1}{2}\left[\mathrm{Tr}\,(X)\,\mathrm{Tr}\,(Y) - \frac{1}{N}\mathrm{Tr}\,(XY)\right]. \end{aligned}$$

From

$$[T^a, T^b] = i f^{abc} T^c$$

one derives by multiplying with T^d and taking the trace:

$$i f^{abc} = 2\left[\mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right)\right]$$

This yields an expression of the structure constants in terms of the matrices of the fundamental representation. We can now calculate for the group $SU(N)$ the fundamental and the adjoint Casimirs:

$$\begin{aligned} \left(T^a T^a\right)_{ij} &= C_F \delta_{ij} = \frac{N^2-1}{2N}\delta_{ij}, \\ f^{abc} f^{dbc} &= C_A \delta^{ad} = N\delta^{ad}. \end{aligned}$$

9.3 Yang-Mills theory

C.N. Yang and R.L.Mills[1] suggested 1954 a generalisation towards non-abelian gauge groups. The field strength tensor is now given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$

where a is an index running from 1 to N^2-1 for a $SU(N)$ gauge group. The Lagrange density reads:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}.$$

The Lagrange density is invariant under the local transformations

$$T^a A^a_\mu(x) \rightarrow U(x)\left(T^a A^a_\mu(x) + \frac{i}{g}\partial_\mu\right) U^\dagger(x)$$

with

$$U(x) = \exp\left(-i T^a \theta_a(x)\right).$$

The action is given by the integral over the Lagrange density:

$$S = \int d^4x\, \mathcal{L}$$

[1]C.N. Yang and R.L. Mills, Phys. Rev. 96, (1954), 191

9.4 Quantisation of gauge theories

Let us examine closer the quantisation of gauge theories. The Lagrange density is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu},$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

This Lagrange density is nicely invariant under local gauge transformations

$$T^a A^a_\mu(x) \rightarrow U(x) \left(T^a A^a_\mu(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x),$$

but we are not yet happy: If we try to calculate the gluon propagator we have to invert a certain matrix and it turns out that this matrix is singular. Something is going wrong. Let´s look again at the generating functional:

$$Z[J] = \int \mathcal{D}A^a_\mu(x) \exp \left[i \int d^4x \, \mathcal{L} + A^a_\mu(x) J^\mu_a(x) \right]$$

The path integral is over all possible gauge field configurations, even the ones which are just related by a gauge transformation. These configuration describe the same physics and it is sufficient to count them only ones. Technically this is done as follows: Let us denote a gauge transformation by

$$U(x) = \exp\left(-i T^b \theta_b(x) \right).$$

The gauge transformation is therefore completely specified by the functions $\theta_b(x)$. We denote by $A^a_\mu(x, \theta_b)$ the gauge field configuration obtained from $A^a_\mu(x)$ through the gauge transforamtion $U(x)$:

$$T^a A^a_\mu(x, \theta_b) = U(x) \left(T^a A^a_\mu(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x),$$

$A^a_\mu(x, \theta_b)$ and $A^a_\mu(x)$ are therefore gauge-equivalent configurations. From all gauge-equivalent configurations we are going to pick the one, which satisfies for a given G^μ and $B^a(x)$ the equation

$$G^\mu A^a_\mu(x, \theta_b) = B^a(x).$$

Let us assume that this equation gives a unique solution θ_b for a given A^a_μ. (This is not necessarily always fulfilled, cases where a unique solution may not exist are known as the Gribov ambiguity.) Let $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$ a n-dimensional vector and let the

$$g_i = g_i(\alpha_1, .., \alpha_n), \quad i = 1, ..., n,$$

be functions of α_j. Then

$$\int \left(\prod_{j=1}^{n} d\alpha_j\right)\left(\prod_{i=1}^{n}\delta\left(g_i(\alpha_1,...,\alpha_n)\right)\right)\det\left(\frac{\partial g_i}{\partial \alpha_j}\right) = 1.$$

Proof: We change the variables from α_j to

$$\beta_i = g_i(\alpha_1,..,\alpha_n).$$

Then

$$\int \left(\prod_{j=1}^{n} d\alpha_j\right)\left(\prod_{i=1}^{n}\delta\left(g_i(\alpha_1,...,\alpha_n)\right)\right)\det\left(\frac{\partial g_i}{\partial \alpha_j}\right) = \int \left(\prod_{j=1}^{n} d\beta_j\right)\left(\prod_{i=1}^{n}\delta\left(\beta_i\right)\right) = 1.$$

We generalise this to the continuum. For a gauge theory with a single generator we obtain:

$$\int \mathcal{D}\theta(x)\,\delta\left(G^\mu A_\mu(x,\theta(x)) - B(x)\right)\det\left(\frac{\delta G^\mu A_\mu(x,\theta(x))}{\delta\theta(y)}\right) = 1.$$

For a gauge theory with n generators we find

$$\int \prod_b \mathcal{D}\theta_b(x)\;\delta^n\left(G^\mu A^a_\mu(x,\theta_b(x)) - B^a(x)\right)\,\det M_G = 1,$$

where

$$\left(M_G(x,y)\right)_{ab} = \frac{\delta G^\mu A^a_\mu(x,\theta_c(x))}{\delta\theta_b(y)}.$$

Remark: θ_b are coordinates of the Lie group:

$$U = \exp\left(-iT^b\theta_b\right)$$

As the Lie group is also a manifold, we can integrate over the manifold. With the coordinates above, the invariant meausure is given by

$$\prod_b \mathcal{D}\theta_b.$$

The integral measure dg is called a left invariant meausure, if

$$\int dg\, f(g_0 g) = \int dg\, f(g)$$

for arbitrary elements g and g_0 of the group G. A meausure is called right invariant, if

$$\int dg\, f(g g_0) = \int dg\, f(g).$$

In general, right and left invariant meausures are not necessarily equal. However, it is known that they are equal for compact groups, simple groups and semi-simple groups.

Remark 2: If the gauge fixing condition is chosen such that

$$G^\mu A^a_\mu(x,\theta_b) - B^a(x) \quad = \quad 0$$

is linear in θ, then the functional derivative

$$\frac{\delta G^\mu A^a_\mu(x,\theta_c(x))}{\delta\theta_b(y)}$$

will be independent of θ and we may take the determinant in front of the integral

$$\det M_G \int \prod_b \mathcal{D}\theta_b(x) \;\; \delta^n\left(G^\mu A^a_\mu(x,\theta_b(x)) - B^a(x)\right) \quad = \quad 1.$$

We now consider

$$Z[0] \quad = \quad \int \mathcal{D}A^a_\mu(x) \; \exp\left[i\int d^4x \; \mathcal{L}(A^a_\mu(x))\right]$$

and insert the gauge-fixing equation

$$\begin{aligned}
Z[0] = & \\
= \;\; & \int \mathcal{D}A^a_\mu(x) \int \prod_b \mathcal{D}\theta_b(x) \; \delta^n\left(G^\mu A^a_\mu(x,\theta_b(x)) - B^a(x)\right) \; \det M_G(A^a_\mu(x,\theta_b(x))) \\
& \times \exp\left[i\int d^4x \; \mathcal{L}(A^a_\mu(x))\right] \\
= \;\; & \int \prod_b \mathcal{D}\theta_b(x) \; \int \mathcal{D}A^a_\mu(x) \det M_G(A^a_\mu(x,\theta_b(x))) \; \delta^n\left(G^\mu A^a_\mu(x,\theta_b(x)) - B^a(x)\right) \\
& \times \exp\left[i\int d^4x \; \mathcal{L}(A^a_\mu(x,\theta_b(x)))\right] \\
= \;\; & \int \prod_b \mathcal{D}\theta_b(x) \; \int \mathcal{D}A^a_\mu(x,\theta_b(x)) \det M_G(A^a_\mu(x,\theta_b(x))) \; \delta^n\left(G^\mu A^a_\mu(x,\theta_b(x)) - B^a(x)\right) \\
& \times \exp\left[i\int d^4x \; \mathcal{L}(A^a_\mu(x,\theta_b(x)))\right] \\
= \;\; & \left(\int \prod_b \mathcal{D}\theta_b(x)\right) \; \int \mathcal{D}A^a_\mu(x) \det M_G(A^a_\mu(x)) \; \delta^n\left(G^\mu A^a_\mu(x) - B^a(x)\right) \exp\left[i\int d^4x \; \mathcal{L}(A^a_\mu(x))\right]
\end{aligned}$$

Here we used the gauge invariance of the action, of $\det M_G$ and of $\mathcal{D}A^a_\mu(x)$. The integral over all gauge-transformations

$$\int \prod_b \mathcal{D}\theta_b(x)$$

is the just an irrelevant prefactor, which we neglect in the following. We then obtain

$$Z[0] = \int \mathcal{D}A^a_\mu(x) \det M_G(A^a_\mu(x))\, \delta^n\left(G^\mu A^a_\mu(x) - B^a(x)\right) \exp\left[i \int d^4x\, \mathcal{L}(A^a_\mu(x))\right]$$

This functional still depends on $B^a(x)$. As we are not interested in any particular choice of $B^a(x)$, we average over $B^a(x)$ with weight

$$\exp\left(-\frac{i}{2\xi}\int d^4x\,(B^a(x)B_a(x))\right)$$

and obtain

$$\begin{aligned}
&\int \mathcal{D}B^a(x)\, Z[0] \exp\left(-\frac{i}{2\xi}\int d^4x\,(B^a(x)B_a(x))\right) = \\
&= \int \mathcal{D}B^a(x) \int \mathcal{D}A^a_\mu(x) \det M_G\, \delta^n\left(G^\mu A^a_\mu(x) - B^a(x)\right) \exp\left[i \int d^4x\, \mathcal{L}(A^a_\mu(x)) - \frac{1}{2\xi}B^a(x)B_a(x)\right] \\
&= \int \mathcal{D}A^a_\mu(x) \det M_G\, \exp\left[i \int d^4x\, \mathcal{L}(A^a_\mu(x)) - \frac{1}{2\xi}\left(G^\mu A^a_\mu(x)\right)(G^\nu A_{\nu\,a}(x))\right].
\end{aligned}$$

We now consider as new generating functional

$$Z[J] = \int \mathcal{D}A^a_\mu(x) \det M_G\, \exp\left[i \int d^4x\, \mathcal{L}(A^a_\mu(x)) - \frac{1}{2\xi}\left(G^\mu A^a_\mu(x)\right)(G^\nu A_{\nu\,a}(x)) + A^a_\mu(x) J^\mu_a(x)\right].$$

We observe that $Z[J]$ contains a term we could expect from a naive fixing of the gauge

$$-\frac{1}{2\xi}\left(G^\mu A^a_\mu(x)\right)(G^\nu A_{\nu\,a}(x)).$$

In addition, the determinant

$$\det M_G$$

appears in front of the exponent.

Various gauges are:

- Lorenz gauge or covariant gauge: $G^\mu = \partial^\mu$.

$$(M_G(x,y))^{ab} = \left(\delta^{ab}\Box - g f^{abc}\partial^\mu A^c_\mu\right)\delta^4(x-y)$$

- Coulomb gauge: $G^\mu = (0, \vec{\nabla})$.

$$(M_G(x,y))^{ab} = \left(\delta^{ab}\nabla^2 - g f^{abc}\vec{A}^c\vec{\nabla}\right)\delta^4(x-y)$$

- Axial gauge: $G^\mu = n^\mu$, where n^μ is a constant four-vector.

$$(M_G(x,y))^{ab} = \left(\delta^{ab} n\cdot\partial - g f^{abc} n\cdot A^c\right)\delta^4(x-y)$$

We would like to exponentiate the determinant. In the treatment of fermions within the path-integral formalism we had the formula

$$\det A \sim \int \mathcal{D}\psi(x)\ \mathcal{D}\bar\psi(x)\ \exp\int d^4x\, d^4y\ \bar\psi(x)A(x,y)\psi(y).$$

We write this as

$$\det A = \int \mathcal{D}\psi(x)\ \mathcal{D}\bar\psi(x)\ \exp -i\int d^4x\, d^4y\ \bar\psi(x)A(x,y)\psi(y).$$

Applying this to $\det M_G$:

$$\det M_G = \int \mathcal{D}c^b(x)\ \mathcal{D}\bar c^a(x)\ \exp\left(i\int d^4x\ \bar c^a(x)\left(-M_G^{ab}\right)c^b(x)\right).$$

Specialising to the covariant gauge $G^\mu = \partial^\mu$ one obtains

$$\begin{aligned}\det M_G &= \int \mathcal{D}c^b(x)\ \mathcal{D}\bar c^a(x)\ \exp\left(i\int d^4x\ \bar c^a(x)\left(-\delta^{ab}\Box + g f^{abc}\partial^\mu A_\mu^c\right)c^b(x)\right)\\ &= \int \mathcal{D}c^b(x)\ \mathcal{D}\bar c^a(x)\ \exp\left(i\int d^4x\ \bar c^a(x)\left(-\partial^\mu D_\mu^{ab}\right)c^b(x)\right),\end{aligned}$$

where

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - g f^{abc} A_\mu^c$$

is the covariant derivative. In the Lorenz gauge we obtain finally

$$\begin{aligned}Z[J,\xi,\bar\xi] = &\int \mathcal{D}A_\mu^a(x)\ \int \mathcal{D}c^b(x)\ \mathcal{D}\bar c^a(x)\\ &\exp\left[i\int d^4x\ \mathcal{L}(A_\mu^a(x)) - \frac{1}{2\xi}\left(\partial^\mu A_\mu^a(x)\right)\left(\partial^\nu A_\nu^a(x)\right) + \bar c^a(x)\left(-\partial^\mu D_\mu^{ab}\right)c^b(x)\right.\\ &\left.+A_\mu^a(x)J_a^\mu(x) + \bar c^a(x)\xi_a(x) + \bar\xi_a(x)c^b(x)\right].\end{aligned}$$

The quantity

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)\left(\partial^\nu A_\nu^a\right)$$

is called the **gauge-fixing term**, the quantity

$$\mathcal{L}_{\text{FP}} = -\bar c^a \partial^\mu D_\mu^{ab} c^b$$

the **Faddeev-Popov term**.

9.5 The Lagrange density for the fermion sector

Example: The Lagrange density for a free electron (e.g. no interactions) is given by

$$\mathcal{L}_F = \bar{\psi}(i\gamma^\mu \partial_\mu - m_e)\psi.$$

We are now looking for a Lagrange density for the fermionic sector, which remains invariant under gauge transformations. Recall that under a gauge transformation a fermion field $\psi_i(x)$ transforms as

$$\begin{aligned} \psi_i(x) &\to U_{ij}(x)\psi_j(x), \qquad U_{ij}(x) = \exp\left(-iT^a\theta^a(x)\right), \\ \bar{\psi}_i(x) &\to \bar{\psi}_j(x)U^\dagger_{ji}(x). \end{aligned}$$

$\theta^a(x)$ depends on the space-time coordinates x. For an infinitessimal gauge transformation we have

$$\psi_i(x) \to \left(1 - iT^a\theta^a(x)\right)\psi_j(x).$$

We immediately see that a fermion mass term

$$-m\bar{\psi}(x)\psi(x)$$

is invariant under gauge transformations. (Note however that in the standard model the fermion masses are generated through the Yukawa couplings to the Higgs field.) But the term involving derivatives is not gauge invariant:

$$i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \to i\bar{\psi}(x)U^\dagger(x)\gamma^\mu\partial_\mu\left(U(x)\psi(x)\right) = i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + \underbrace{i\bar{\psi}(x)\gamma^\mu\left(U^\dagger(x)\partial_\mu U(x)\right)\psi(x)}_{\text{extra}}.$$

The solution comes in the form of the covariant derivative

$$D_\mu = \partial_\mu - igT^aA^a_\mu(x),$$

where the gauge field transforms as

$$T^aA^a_\mu(x) \to U(x)\left(T^aA^a_\mu(x) + \frac{i}{g}\partial_\mu\right)U^\dagger(x).$$

Then the combination

$$\begin{aligned} i\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) &= i\bar{\psi}(x)\gamma^\mu\left(\partial_\mu - igT^aA^a_\mu(x)\right)\psi(x) \\ &\to i\bar{\psi}(x)U^\dagger(x)\gamma^\mu\left[\partial_\mu - igU(x)\left(T^aA^a_\mu(x) + \frac{i}{g}\partial_\mu\right)U^\dagger(x)\right]U(x)\psi(x) \\ &= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + i\bar{\psi}(x)\gamma^\mu\left(U^\dagger(x)\partial_\mu U(x)\right)\psi(x) \\ &\quad + g\bar{\psi}(x)\gamma^\mu T^aA^a_\mu(x)\psi(x) + i\bar{\psi}(x)\gamma^\mu\left[\left(\partial_\mu U^\dagger(x)\right)U(x)\right]\psi(x) \end{aligned}$$

is invariant.

The Lagrange density for the fermion sector:

$$\mathcal{L}_{\text{fermions}} = \sum_{\text{fermions}} \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x).$$

9.6 Feynman rules for QED and QCD

Expanding the Lagrange density into terms bilinear in the fields and interaction terms. As an example consider the gluonic part of the Lagrange density:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 + \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{fermions}}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

This yields

$$\begin{aligned}\mathcal{L}_{\text{QCD}} = & -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \\ & - g f^{abc}\left(\partial_\mu A^a_\nu\right) A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 \left(f^{eab} A^a_\mu A^b_\nu\right)\left(f^{ecd} A^{c\mu} A^{d\nu}\right) \\ & + \mathcal{L}_{\text{FP}} + \mathcal{L}_{\text{fermions}}.\end{aligned}$$

Terms bilinear in the fields define the propagators. Terms with three or more fields define interaction vertices.
For QED (photons and electrons) the Lagrange density is given by

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\partial\!\!\!/ - m_e)\psi - \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + e\bar{\psi}\gamma^\mu A_\mu \psi.$$

In the QED case, the Faddeey-Popov ghosts have no interactions and behave as non-interacting free particles. We may ignore them in the QED case.

9.6.1 Propagators

Terms bilinear in the fields yield propagators. Consider

$$\begin{aligned}&\mathcal{L}_{\text{bilinear}}(x) = \\ &\quad \frac{1}{2}\phi_i(x) P^{\text{boson,real}}_{ij}(x)\phi_j(x) + \chi^*_i(x) P^{\text{boson,complex}}_{ij}(x)\chi_j(x) + \bar{\psi}_i(x) P^{\text{fermion}}_{ij}(x)\psi_j(x),\end{aligned}$$

where ϕ_i denotes a set of real boson fields (one degree of freedom), χ_i denotes a set of complex boson fields (two degrees of freedom), and ψ_i denotes a set of fermion fields. The boson fields may be scalar or vector fields. P is a matrix operator that may contain derivatives and must have an inverse. P is taken to be a hermitian operator (real symmetric operator in the case of real boson fields):

$$P^\dagger = P$$

Define the inverse of P by

$$\sum_j P_{ij}(x) P^{-1}_{jl}(x-y) \;=\; \delta_{il}\delta^4(x-y),$$

and its Fourier transform by

$$P^{-1}_{ij}(x) \;=\; \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{P}^{-1}_{ij}(k).$$

Then the propagator is given by

$$\Delta_F(k)_{ij} \;=\; \int d^4x e^{ik\cdot x} \langle 0|T(\phi_i(x)\phi_j(0)|0\rangle = i\left(\tilde{P}^{-1}(k)\right)_{ij}.$$

The propagator of a scalar particle:

From

$$P(x) \;=\; -\Box - m^2$$

we found already

$$\tilde{P}^{-1}(k) \;=\; \frac{1}{k^2-m^2}$$

and therefore

$$\Delta_F(k) \;=\; \frac{i}{k^2-m^2}.$$

The propagator of a fermion: We start from

$$P(x) \;=\; i\partial\!\!\!/ - m.$$

Then

$$(i\partial\!\!\!/_x - m)\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \tilde{P}^{-1}(k) = \int \frac{d^4k}{(2\pi)^4} (k\!\!\!/ - m)\, e^{-ik\cdot(x-y)} \tilde{P}^{-1}(k) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}.$$

Therefore

$$(k\!\!\!/ - m)\,\tilde{P}^{-1}(k) \;=\; 1$$

and

$$\Delta_F(k) \;=\; i\frac{k\!\!\!/ + m}{k^2-m^2}.$$

The propagator of a gauge boson:

In Lorentz gauge the gauge-fixing term is given by

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

and therefore

$$P(x) = \Box g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu.$$

Then

$$P(x)\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}\tilde{P}^{-1}(k) = \int \frac{d^4k}{(2\pi)^4} k^2 \left(-g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)\frac{k^\mu k^\nu}{k^2}\right) e^{-ik\cdot(x-y)}\tilde{P}^{-1}(k).$$

We have for

$$M_{\mu\nu} = -g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)\frac{k_\mu k_\nu}{k^2} \quad \text{and} \quad N_{\mu\nu} = -g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}$$

the following relation:

$$M_{\mu\lambda}N^{\lambda\nu} = g_\mu{}^\nu.$$

Therefore

$$\Delta_F(k) = \frac{i}{k^2}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right).$$

9.6.2 Vertices

A general term in $\mathcal{L}_{\text{int}}(x)$ has the form

$$\mathcal{L}_{\text{int}}(x) = \int d^4x_1 ... d^4x_n \alpha_{i_1...i_n}(x, x_1, ..., x_n)\phi_{i_1}(x_1)...\phi_{i_n}(x_n).$$

For the vertex we define

$$\alpha_{i_1...i_n}(x, x_1, ..., x_n) = \int \frac{d^4k_1}{(2\pi)^4} ... \frac{d^4k_n}{(2\pi)^4} e^{-ik_1(x-x_1)-...-ik_n(x-x_n)}\tilde{\alpha}_{i_1...i_n}(k_1, ..., k_n).$$

$\tilde{\alpha}$ contains a factor ik_{j_μ} for every derivative $\partial/\partial x_{j_\mu}$ acting on a field with argument x_j. The vertex is then given by

$$I(k_1, ..., k_n) = i \sum_{\text{permutations}} (-1)^P \tilde{\alpha}_{i_1...i_n}(k_1, ..., k_n).$$

The summations are over all permutations of indices and momenta. The momenta are taken to flow inward.

Example: The quark-gluon vertex

$$\begin{aligned}\mathcal{L}_{\text{int}} &= g\bar{\psi}^i\gamma^\mu(T^a)^{ij}A^a_\mu\psi^j = \int d^4x_1\int d^4x_2\int d^4x_3\delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\\ &\quad\times\left[g\gamma^\mu(T^a)^{ij}\right]\bar{\psi}^i(x_1)\psi^j(x_2)A^a_\mu(x_3).\end{aligned}$$

Then

$$\tilde{\alpha}(k_1,k_2,k_3) = g\gamma^\mu(T^a)^{ij}$$

and

$$I = ig\gamma^\mu(T^a)^{ij}.$$

Example 2: The three-gluon vertex

$$\begin{aligned}&\mathcal{L}_{\text{int}} = -gf^{abc}\left(\partial_\mu A^a_\nu\right)A^{b\mu}A^{c\nu}\\ &= \int d^4x_1\int d^4x_2\int d^4x_3\delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\left(-gf^{abc}\right)\partial^\nu_{x_1}g^{\mu\lambda}A^a_\mu(x_1)A^b_\nu(x_2)A^c_\lambda(x_3)\\ &= \int d^4x_1\int d^4x_2\int d^4x_3A^a_\mu(x_1)A^b_\nu(x_2)A^c_\lambda(x_3)gf^{abc}g^{\mu\lambda}\partial^\nu_{x_1}\delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3).\end{aligned}$$

Therefore

$$\begin{aligned}\alpha(x,x_1,x_2,x_3) &= gf^{abc}g^{\mu\lambda}\partial^\nu_{x_1}\delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\\ &= gf^{abc}g^{\mu\lambda}\partial^\nu_{x_1}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}e^{-ik_1(x-x_1)}e^{-ik_2(x-x_2)}e^{-ik_3(x-x_3)}\\ &= \int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}gf^{abc}g^{\mu\lambda}ik_1^\nu e^{-ik_1(x-x_1)}e^{-ik_2(x-x_2)}e^{-ik_3(x-x_3)}\end{aligned}$$

and

$$\tilde{\alpha}(k_1,k_2,k_3) = gf^{abc}g^{\mu\lambda}ik_1^\nu.$$

Then

$$I = i\sum_{\text{permutations}}\tilde{\alpha}(k_1,k_2,k_3) = gf^{abc}\left[(k_2-k_3)_\mu g_{\nu\lambda}+(k_3-k_1)_\nu g_{\lambda\mu}+(k_1-k_2)_\lambda g_{\mu\nu}\right].$$

9.6.3 List of Feynman rules

Propagators:

The propagators for the gauge bosons are in the Feynman gauge ($\xi = 1$).

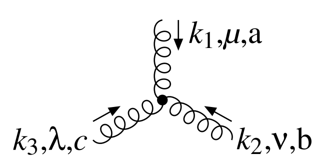
gauge bosons	gluon	A_μ^a	$\frac{-ig_{\mu\nu}}{k^2}\delta_{ab}$
	photon	A_μ	$\frac{-ig_{\mu\nu}}{k^2}$
fermions	quarks	ψ_i	$i\frac{\not{p}+m}{p^2-m^2}\delta_{ij}$
	leptons	ψ	$i\frac{\not{p}+m}{p^2-m^2}$
ghosts		c^a	$\frac{i}{k^2}\delta^{ab}$

Vertices:

Quark-gluon-vertex:

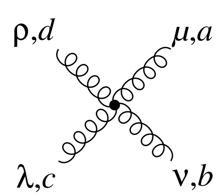
$$ig\gamma_\mu T_{ij}^a$$

3-gluon-vertex:



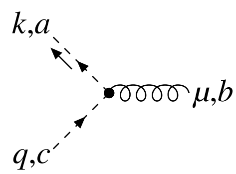
$$gf^{abc}\left[(k_2-k_3)_\mu g_{\nu\lambda}+(k_3-k_1)_\nu g_{\lambda\mu}+(k_1-k_2)_\lambda g_{\mu\nu}\right]$$

4-gluon-vertex:



$$-ig^2\left[f^{abe}f^{ecd}\left(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}\right)+f^{ace}f^{ebd}\left(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\lambda\nu}\right)+f^{ade}f^{ebc}\left(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}\right)\right]$$

Gluon-ghost-vertex:



$$-gf^{abc}k_\mu$$

Fermion-photon-vertex:

$$ieQ\gamma_\mu$$

Additional rules:

An integration

$$\int\frac{d^4k}{(2\pi)^4}$$

for each loop.

A factor (-1) for each closed fermion loop.

Symmetry factor: Multiply the diagram by a factor $1/S$, where S is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed.

External particles:

Outgoing fermion: $\bar{u}(p)$
Outgoing antifermion: $v(p)$

Incoming fermion: $u(p)$
Incoming antifermion: $\bar{v}(p)$

Gauge boson: $\varepsilon_\mu(k)$

Polarisation sums:

$$\begin{aligned}\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) &= p\!\!\!/ + m,\\ \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) &= p\!\!\!/ - m,\end{aligned}$$

$$\sum_\lambda \varepsilon^*_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) = -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{kn} - n^2\frac{k_\mu k_\nu}{(kn)^2}.$$

Here n^μ is an arbitrary four vector. The dependence on n^μ cancels in gauge-invariant quantities. Using Weyl spinors, a convenient choice of polarisation vectors for the gauge bosons is given by

$$\begin{aligned}\varepsilon^+_\mu(k,q) &= \frac{\langle q-|\gamma_\mu|k-\rangle}{\sqrt{2}\langle qk\rangle},\\ \varepsilon^-_\mu(k,q) &= \frac{\langle q+|\gamma_\mu|k+\rangle}{\sqrt{2}\,[kq]},\end{aligned}$$

where q^μ is an arbitrary light-like reference momentum. The dependence on q^μ cancels in gauge-invariant quantities. The polarisation vectors satisfy:

$$\begin{aligned}\varepsilon^\pm_\mu(k,q)k^\mu &= 0,\\ \varepsilon^\pm_\mu(k,q)q^\mu &= 0.\end{aligned}$$

$$\begin{aligned}\varepsilon^+\cdot\left(\varepsilon^+\right)^* &= \varepsilon^-\cdot\left(\varepsilon^-\right)^* = -1,\\ \varepsilon^+\cdot\left(\varepsilon^-\right)^* &= 0.\end{aligned}$$

$$\left(\varepsilon^+_\mu\right)^* = \varepsilon^-_\mu.$$

9.7 Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors (carrying the colour structures) multiplied by kinematic functions called partial amplitudes. These partial amplitudes do not contain any colour information and are gauge-invariant objects. Partial amplitudes may be decomposed further into primitive amplitudes.

The colour decomposition is obtained by replacing the structure constants f^{abc} by

$$if^{abc} = 2\left[\mathrm{Tr}\left(T^aT^bT^c\right) - \mathrm{Tr}\left(T^bT^aT^c\right)\right]$$

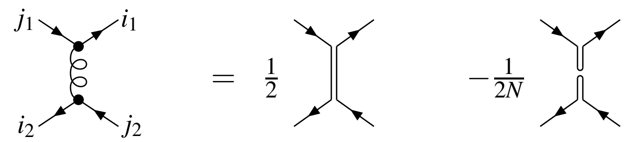
which follows from $\left[T^a, T^b\right] = i f^{abc} T^c$. The resulting traces and strings of colour matrices can be further simplified with the help of the Fierz identity :

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right).$$

As an example one finds

$$\begin{aligned} i f^{a_1 a_2 b} i f^{b a_3 a_4} &= 4\left[\mathrm{Tr}\left(T^{a_1}T^{a_2}T^b\right) - \mathrm{Tr}\left(T^{a_2}T^{a_1}T^b\right)\right]\left[\mathrm{Tr}\left(T^{a_3}T^{a_4}T^b\right) - \mathrm{Tr}\left(T^{a_4}T^{a_3}T^b\right)\right] \\ &= 2\,\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) - 2\,\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_4}T^{a_3}\right) - 2\,\mathrm{Tr}\left(T^{a_2}T^{a_1}T^{a_3}T^{a_4}\right) \\ &\quad +2\,\mathrm{Tr}\left(T^{a_2}T^{a_1}T^{a_4}T^{a_3}\right). \end{aligned}$$

The colour algebra can be carried out diagrammatically, resulting in colour flow lines. As an example we consider the exchange of a gluon between two quarks. Concentrating only on the colour part we can use the double line notation of 't Hooft and write symbolically:



$$T^a_{i_1 j_1} T^a_{i_2 j_2} = \tfrac{1}{2}\delta_{i_1 j_2}\delta_{i_2 j_1} - \tfrac{1}{2N}\delta_{i_1 j_1}\delta_{i_2 j_2}$$

In the last line we have used the Fierz identity to contract out the generators of the $SU(3)$ algebra. In the pure Yang-Mills case tree amplitudes with n external gluons may be written in the form

$$\mathcal{A}_n^{(0)}\left(g_1, g_2, ..., g_n\right) = g^{n-2} \sum_{\sigma \in S_n/Z_n} 2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)\; A_n^{(0)}\left(g_{\sigma(1)}, ..., g_{\sigma(n)}\right),$$

where the sum is over all non-cyclic permutations of $\{1, 2, ..., n\}$. The quantities $A_n^{(0)}(g_{\sigma(1)}, ..., g_{\sigma(n)})$ accompanying the colour factor $2\,\mathrm{Tr}(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}})$ are called **partial amplitudes**. Partial amplitudes are gauge-invariant and are defined as the kinematic coefficients of the independent colour structures. Closely related are **primitive amplitudes**, which for tree-level Yang-Mills amplitudes are calculated from planar diagrams with a fixed cyclic ordering of the external legs and cyclic-ordered Feynman rules. Primitive amplitudes are gauge invariant as well. For tree-level Yang-Mills amplitudes the notions of partial amplitudes and primitive amplitudes coincide. However, this is no longer true if one considers amplitudes with quarks and/or amplitudes with loops. The most important features of a primitive amplitude are gauge invariance and a fixed cyclic ordering of the external legs. In particular they can only have singularities like poles and cuts in a limited set of momenta channels, those made out of sums of cyclically adjacent momenta. (For amplitudes with quarks and/or loops there will be some additional requirements, which are not relevant here.) Partial amplitudes are also gauge invariant, but not necessarily cyclic ordered. The leading contributions in an $1/N$-expansion (with N being the number of colours) are usually cyclic ordered, the sub-leading parts are in general not.

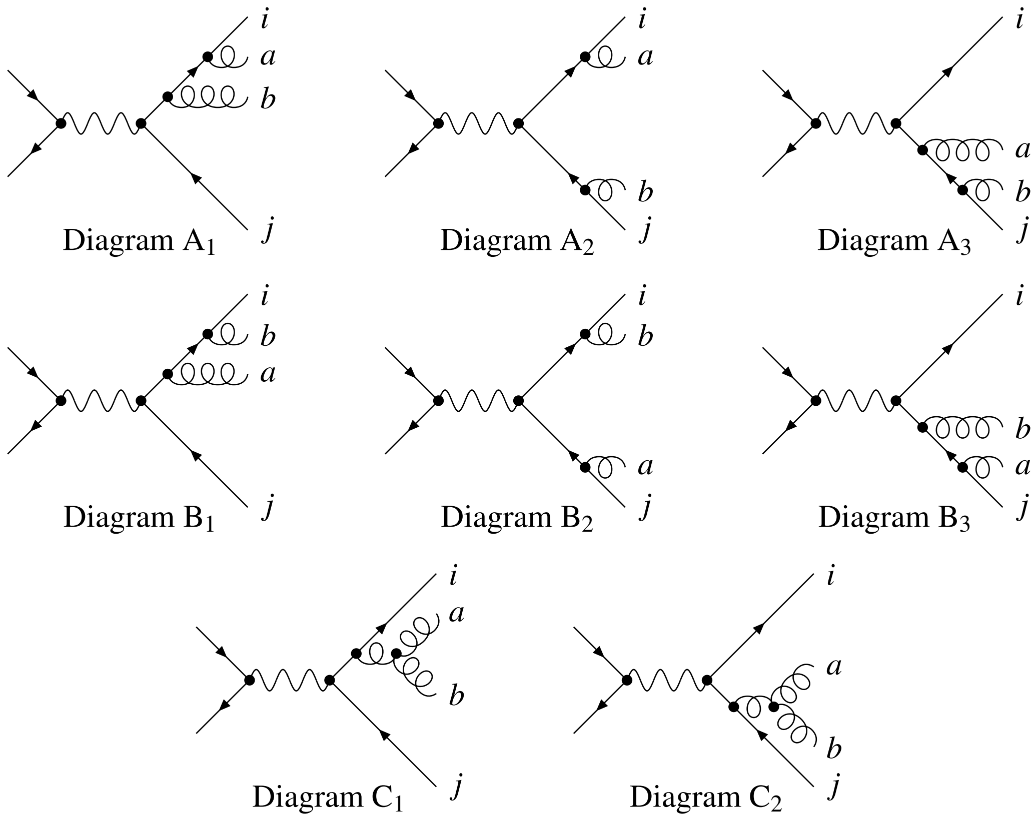


Figure 1: Diagrams contributing to $e^+e^- \to qgg\bar{q}$.

The colour decomposition for a tree amplitude with a pair of quarks is

$$\mathcal{A}_n^{(0)}(q, g_1, ..., g_{n-2}, \bar{q}) \;=\; g^{n-2} \sum_{S_{n-2}} \left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n-2)}} \right)_{i_q j_{\bar{q}}} A_n^{(0)}(q, g_{\sigma(1)}, ..., g_{\sigma(n-2)}, \bar{q}),$$

where the sum is over all permutations of the gluon legs. Similar decompositions may be obtained for amplitudes with more than one pair of quarks and/or amplitudes with electroweak particles.

Let us consider an example: $e^+e^- \to qgg\bar{q}$. In fig. (1) we show all tree diagrams contributing to $\mathcal{A}_6^{(0)}(e^+, e^-, q, g_1, g_2, \bar{q})$ via photon exchange. We have the colour decomposition

$$\mathcal{A}_6^{(0)}\left(e^+, e^-, q, g_1, g_2, \bar{q}\right) =$$
$$e^2 g^2 \left[\left(T^a T^b\right)_{ij} A_6^{(0)}(e^+, e^-, q, g_1, g_2, \bar{q}) + \left(T^b T^a\right)_{ij} A_6^{(0)}(e^+, e^-, q, g_2, g_1, \bar{q}) \right].$$

Let us now discuss, which diagrams contribute to the individual partial amplitudes. We group the diagrams into three classes, A, B and C, as shown in fig. (1).

Colour factor of diagrams of class A_j:

$$\left(T^a T^b\right)_{ij}$$

Colour factor of diagrams of class B_j:

$$\left(T^b T^a\right)_{ij}$$

Colour factor of diagrams of class C_j:

$$\begin{aligned} T^c_{ij} i f^{abc} &= 2 T^c_{ij} \left(\text{Tr}\, T^a T^b T^c - \text{Tr}\, T^b T^a T^c \right) \\ &= \left(T^a T^b\right)_{ij} - \left(T^b T^a\right)_{ij} \end{aligned}$$

Note that the third colour structure is a linear combination of the first two and diagrams from class C contribute to both partial amplitudes.

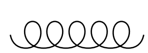
The two colour factors $(T^a T^b)_{ij}$ and $(T^b T^a)_{ij}$ are orthogonal to leading order in $1/N$:

$$\begin{aligned} \left(T^a_{ik} T^b_{kj}\right)^\dagger \left(T^a_{ik} T^b_{kj}\right) &= \text{Tr}\, T^a T^a T^b T^b = \left(\frac{1}{2}\right)^2 \frac{(N^2-1)^2}{N} = \frac{1}{4} N^3 + O(N) \\ \left(T^a_{ik} T^b_{kj}\right)^\dagger \left(T^b_{ik} T^a_{kj}\right) &= \text{Tr}\, T^a T^b T^a T^b = -\left(\frac{1}{2}\right)^2 \frac{N^2-1}{N} = O(N) \end{aligned}$$

Orthogonality to leading order in $1/N$ can be used to prove that the partial amplitudes are individually gauge invariant: The full amplitude $\mathcal{A}_n^{(0)}$ is gauge invariant and the partial amplitudes $A_n^{(0)}$ do not depend on N, therefore multiplying the colour decomposition by an appropriate string of colour matrices and taking the $N \to \infty$-limit shows that the individual partial amplitudes are gauge invariant.

We already mentioned that for tree-level amplitudes with zero or one quark-antiquark pair the notions of partial amplitudes and primitive amplitudes coincide. Primitive amplitudes are calculated from planar diagrams with a fixed cyclic ordering of the external legs and cyclic-ordered Feynman rules. The cyclic-ordered Feynman rules are obtained from the standard Feynman rules by extracting from each formula the coupling constant and by taking the coefficient of the cyclic-ordered colour part. Let us now list the **cyclic-ordered Feynman rules**. The propagators for quark, gluon and ghost particles are given by

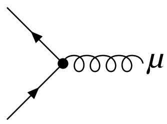
$$= i \frac{p\!\!\!/ + m}{p^2 - m^2},$$



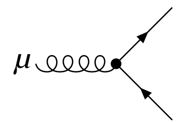
$$= \frac{-i g^{\mu\nu}}{p^2},$$

$$= \frac{i}{p^2}.$$

The gluon prpagator is given in Feynman gauge. The Feynman rule for the quark-gluon vertex is given by

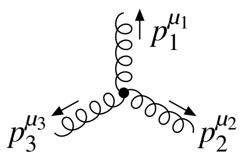


$$= i\gamma^\mu,$$

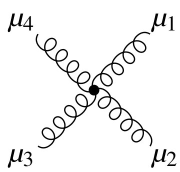


$$= -i\gamma^\mu.$$

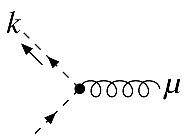
The cyclic-ordered Feynman rules for the three-gluon vertex, the four-gluon vertex and the gluon-ghost vertex are



$$= i\left[g^{\mu_1\mu_2}\left(p_1^{\mu_3}-p_2^{\mu_3}\right)+g^{\mu_2\mu_3}\left(p_2^{\mu_1}-p_3^{\mu_1}\right)+g^{\mu_3\mu_1}\left(p_3^{\mu_2}-p_1^{\mu_2}\right)\right],$$



$$= i\left[2g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right],$$



$$= ik_\mu.$$

Note that the Feynman rule for the cyclic-ordered four-gluon vertex is considerably simpler than the Feynman rule for the full four-gluon vertex.

In addition, there are fewer diagrams contributing to a cyclic-ordered primitive amplitude $A_n^{(0)}$ than to the full amplitude $\mathcal{A}_n^{(0)}$. For the all-gluon tree amplitudes this is illustrated in the following table, giving the number of diagrams contributing to the full amplitude $\mathcal{A}_n^{(0)}$ and to the cyclic-ordered primitive amplitude $A_n^{(0)}$:

n	4	5	6	7	8	9	10
unordered	4	25	220	2485	34300	559405	10525900
cyclic ordered	3	10	38	154	654	2871	12925

The number of diagrams contributing to the cyclic-ordered primitive amplitude $A_n^{(0)}$ is significantly smaller than the number of Feynman diagrams contributing to the full amplitude $\mathcal{A}_n^{(0)}$.

10 Example calculations for scattering amplitudes

Let us go through a few examples for the calculation of tree-level scattering amplitudes.

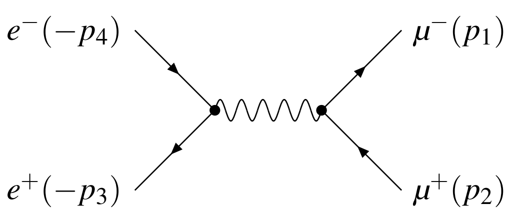
10.1 Examples from QED: Bhabha, Møller and annihilation

Let us start with three examples from QED: Bhabha scattering, Møller scattering and electron positron annihilation into a a pair of muons. Thus, we consider the following processes:

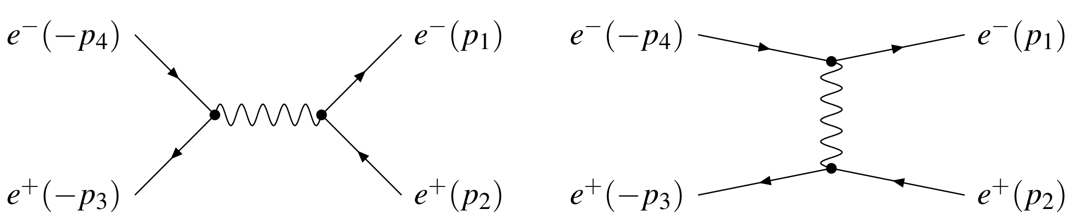
Annihilation:	$e^+e^- \to \mu^+\mu^-$
Bhabha scattering:	$e^+e^- \to e^+e^-$
Møller scattering:	$e^-e^- \to e^-e^-$

In this section we discuss the traditional text-book method for the computation of the amplitude and the amplitude squared. This method is acceptable for simple processes, but soon runs out of steam for more complicated processes. In the next section we will discuss advanced methods based on spinor helicity techniques and colour decomposition. The relevant Feynman graphs are:

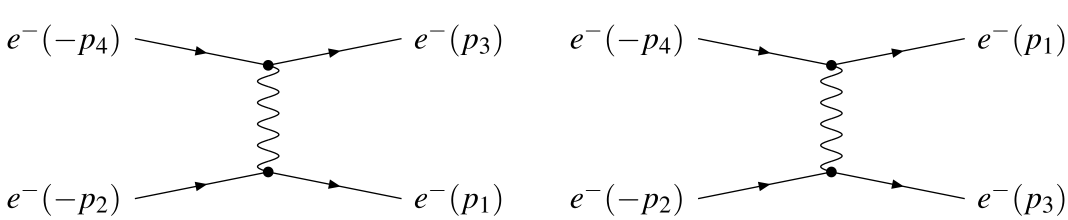
Annihilation:



Bhabha:



Møller:



Let us start with the lowest order amplitude for the annihilation process. We denote this amplitude by

$$\mathcal{A}(p_1, p_2, p_3, p_4).$$

It is convenient to take all momenta as outgoing, e.g. to consider the process

$$\emptyset \quad \to \quad p_1 + p_2 + p_3 + p_4.$$

Energy-momentum conservation reads then

$$p_1 + p_2 + p_3 + p_4 \quad = \quad 0.$$

At high energies we may neglect the fermion masses. Then there is no difference between the diagram for the annihilation process and the first diagram for Bhabha scattering. The second diagram for Bhabha scattering is obtained by exchanging $2 \leftrightarrow 4$. Therefore we have

$$\mathcal{A}_{\text{Bhabha}} \quad = \quad \mathcal{A}(p_1, p_2, p_3, p_4) - \mathcal{A}(p_1, p_4, p_3, p_1)$$

The minus sign comes from the exchange of two fermions. p_1 and p_2 are the momenta of the final state particles, $-p_3$ and $-p_4$ are the momenta of the initial state particles. For Møller scattering one has the same formula:

$$\mathcal{A}_{\text{Moller}} \quad = \quad \mathcal{A}(p_1, p_2, p_3, p_4) - \mathcal{A}(p_1, p_4, p_3, p_1),$$

the difference corresponds only to the assignment of initial and final state momenta: For Møller scattering p_1 and p_3 denote the final state momenta, $-p_2$ and $-p_4$ denote the initial state momenta.

Let us now calculate the amplitude for the annihilation process. Using the Feynman rules we obtain

$$\begin{aligned}
\mathcal{A}(p_1, p_2, p_3, p_4) &= \bar{u}(p_1)\,(-ie\gamma^\mu)\,v(p_2)\ \ \frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\ \ \bar{u}(p_3)\,(-ie\gamma^\nu)\,v(p_4) \\
&= \frac{ie^2}{(p_1+p_2)^2}\left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]
\end{aligned}$$

What we actually need is the amplitude squared, summed over all spins:

$$\begin{aligned}
&\sum_{\text{spin}} |\mathcal{A}|^2 = \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \left|\frac{ie^2}{(p_1+p_2)^2}\right|^2 \left(\left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]\right)\left(\left[\bar{u}(p_1)\gamma^\nu v(p_2)\right]\left[\bar{u}(p_3)\gamma_\nu v(p_4)\right]\right)^* \\
&= \frac{e^4}{s_{12}^2} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]\left[\bar{v}(p_4)\gamma_\nu u(p_3)\right] \\
&= \frac{e^4}{s_{12}^2} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]\left[\bar{v}(p_4)\gamma_\nu u(p_3)\right]
\end{aligned}$$

We can now use the polarisation sums

$$\sum_{\lambda} u(p)\bar{u}(p) = \sum_{\lambda} u(p)\bar{u}(p) = \not{p}$$

and obtain

$$\sum_{\text{spin}} |\mathcal{A}|^2 = \frac{e^4}{s_{12}^2} \left(\text{Tr}\, \not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu\right) \left(\text{Tr}\, \not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu\right)$$

We now have to work out the traces over the Dirac matrices. We recall

$$\begin{aligned}
\text{Tr}\, \gamma^\mu \gamma^\nu &= 4 g^{\mu\nu}, \\
\text{Tr}\, \gamma^{\mu_1} \gamma^{\mu_2} ... \gamma^{\mu_{2n}} &= g^{\mu_1\mu_2} \text{Tr}\, \gamma^{\mu_3} ... \gamma^{\mu_{2n}} - g^{\mu_1\mu_3} \text{Tr}\, \gamma^{\mu_2} \gamma^{\mu_4} ... \gamma^{\mu_{2n}} + g^{\mu_1\mu_4} \text{Tr}\, \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_5} ... \gamma^{\mu_{2n}} - ..., \\
\text{Tr}\, \gamma^{\mu_1} \gamma^{\mu_2} ... \gamma^{\mu_{2n-1}} &= 0, \\
\text{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 &= 4 i \varepsilon^{\mu\nu\rho\sigma}.
\end{aligned}$$

With the help of these theorems we find

$$\text{Tr}\, \not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu = 2 s_{12} \left(-g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{p_1 \cdot p_2} \right)$$

and therefore

$$\begin{aligned}
\sum_{\text{spin}} |\mathcal{A}|^2 &= 4 e^4 \left(-g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu}{p_1 \cdot p_2} \right) \left(-g_{\mu\nu} + \frac{p_{3\,\mu} p_{4\,\nu} + p_{4\,\mu} p_{3\,\nu}}{p_3 \cdot p_4} \right) \\
&= \frac{16 e^4}{s_{12}^2} \left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu \right) \left(p_{3\,\mu} p_{4\,\nu} + p_{4\,\mu} p_{3\,\nu} \right) \\
&= 8 e^4 \frac{s_{13} s_{24} + s_{14} s_{23}}{s_{12}^2},
\end{aligned}$$

with $s_{ij} = (p_i + p_j)^2$. Let us introduce the Mandelstam variables

$$\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\
t &= (p_2 + p_3)^2 = (p_1 + p_4)^2, \\
u &= (p_1 + p_3)^2 = (p_2 + p_4)^2.
\end{aligned}$$

For massless external particles we have

$$s + t + u = 0.$$

Our final result for the annihilation process reads

$$\sum_{\text{spin}} |\mathcal{A}|^2 = 8 e^4 \frac{t^2 + u^2}{s^2}.$$

10.2 Calculation of helicity amplitudes

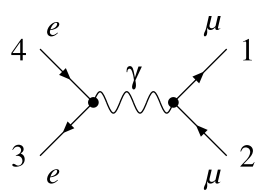
The method discussed in the previous section has the following drawback: Assume that $\mathcal{A}$ is given by N_{terms} terms. N_{terms} can be a rather large number. Squaring the amplitude and summing over the spins will result in $O(N_{\text{terms}}^2)$ terms. It is more efficient to use explicit representations for the polarisation states, calculate the amplitude for each polarisation configuration (this is an $O(N_{\text{terms}})$ operation), square the amplitude (for a given polarisation configuration this is an $O(1)$ operation, basically computing the norm of a complex number) and then summing over all polarisation configurations. If each of the n external particles has two polarisation states, the computational cost is

$$2^n N_{\text{terms}},$$

which for large N_{terms} is much smaller than N_{terms}^2. This is the idea of the spinor helicity method.

Example 1: $e^+e^- \to \mu^+\mu^-$

We first reconsider the annihilation process $e^+e^- \to \mu^+\mu^-$.



As before we use the convention that all momenta are outgoing. We now calculate individual helicity amplitudes, which depend on the set of external momenta $\{p_1, p_2, p_3, p_4\}$ and a set of helicities $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. It is convenient to introduce the short-hand notation

$$\mathcal{A}\left(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}\right) = \mathcal{A}\left(p_1, \lambda_1, p_2, \lambda_2, p_3, \lambda_3, p_4, \lambda_4\right).$$

The Feynman rules for the polarisations of the external massless fermions are

Outgoing fermion with positive helicity:	$\langle p+\vert$
Outgoing fermion with negative helicity:	$\langle p-\vert$
Outgoing antifermion with positive helicity:	$\vert p-\rangle$
Outgoing antifermion with negative helicity:	$\vert p+\rangle$

For the process $e^+e^- \to \mu^+\mu^-$ there are no colour charges and therefore the "colour decomposition" is trivial, we simply factor out the couplings:

$$\mathcal{A}\left(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}\right) = (-e)^2 A\left(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}\right).$$

We have 4 external particles and each external particle has two spin or helicity states. Therefore we have a priori $2^4 = 16$ helicity amplitudes. However, some of them are zero. This is due to the fact that for massless fermions helicity is conserved. Let us look how a massless fermion couples to a gauge boson:

$$(\langle p_1 - |, \langle p_1 + |) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} |p_2+\rangle \\ |p_2-\rangle \end{pmatrix}.$$

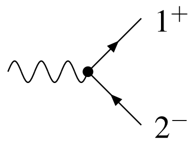
The only non-zero contributions are

$$\langle p_1 - |\sigma^\mu|p_2-\rangle \quad \text{and} \quad \langle p_1 + |\bar{\sigma}^\mu|p_2+\rangle.$$

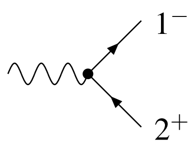
Therefore we have only $2^2 = 4$ non-zero helicity amplitudes:

$$A(p_1^+, p_2^-, p_3^+, p_4^-), \quad A(p_1^+, p_2^-, p_3^-, p_4^+), \quad A(p_1^-, p_2^+, p_3^+, p_4^-), \quad A(p_1^-, p_2^+, p_3^-, p_4^+).$$

In addition, there are relations between the non-zero helicity amplitudes. The muon pair couples through



$$= \langle 1 + |\gamma^\mu| 2+\rangle,$$

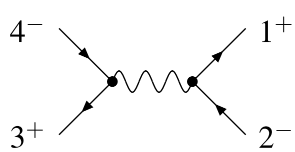


$$= \langle 1 - |\gamma^\mu| 2-\rangle = \langle 2 + |\gamma^\mu| 1+\rangle.$$

This implies that the helicity configuration p_1^-, p_2^+ can be obtained from the helicity configuration p_1^+, p_2^- by exchanging $p_1 \leftrightarrow p_2$. The same holds true for the electron pair: The helicity configuration p_3^-, p_4^+ can be obtained from the helicity configuration p_3^+, p_4^- by exchanging $p_3 \leftrightarrow p_4$. It follows that only one helicity amplitude needs to be calculated, which can be taken to be

$$A(p_1^+, p_2^-, p_3^+, p_4^-).$$

Let us now calculate $A(p_1^+, p_2^-, p_3^+, p_4^-)$:



$$\begin{aligned} A(p_1^+, p_2^-, p_3^+, p_4^-) &= \left\langle 1 + \left| i\gamma_\mu \right| 2+ \right\rangle \frac{-ig^{\mu\nu}}{(p_1+p_2)^2} \langle 3 + |i\gamma_\nu| 4+\rangle \\ &= \frac{i}{s_{12}} \left\langle 1 + \left| \gamma_\mu \right| 2+ \right\rangle \langle 3 + |\gamma^\mu| 4+\rangle \\ &= \frac{2i}{s_{12}} [13]\langle 42\rangle. \end{aligned}$$

The other non-zero helicity amplitudes are then given by

$$
\begin{aligned}
A(p_1^+, p_2^-, p_3^-, p_4^+) &= \frac{2i}{s_{12}}[14]\langle 32\rangle, \\
A(p_1^-, p_2^+, p_3^+, p_4^-) &= \frac{2i}{s_{12}}[23]\langle 41\rangle, \\
A(p_1^-, p_2^+, p_3^-, p_4^+) &= \frac{2i}{s_{12}}[24]\langle 31\rangle.
\end{aligned}
$$

We note that since $[13]\langle 34\rangle = -[12]\langle 24\rangle$ we may equally write

$$
A(p_1^+, p_2^-, p_3^+, p_4^-) = 2i\frac{[13]\langle 42\rangle}{s_{12}} = 2i\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 43\rangle}.
$$

Squaring the amplitude one obtains

$$
\left|A(p_1^+, p_2^-, p_3^+, p_4^-)\right|^2 = \frac{4 s_{13} s_{24}}{s_{12}^2} = \frac{4u^2}{s^2},
$$

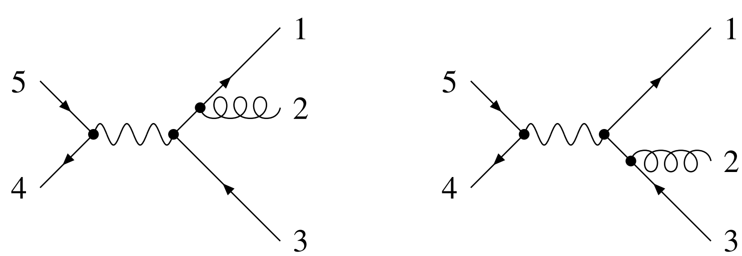
and similar for the other helicity amplitudes. Summing over all helicities one finally obtains

$$
\begin{aligned}
\sum_{\text{spin}} |\mathcal{A}|^2 &= e^4 \left(\left|A(p_1^+, p_2^-, p_3^+, p_4^-)\right|^2 + \left|A(p_1^+, p_2^-, p_3^-, p_4^+)\right|^2 + \left|A(p_1^-, p_2^+, p_3^+, p_4^-)\right|^2 \right. \\
&\quad \left. + \left|A(p_1^-, p_2^+, p_3^-, p_4^+)\right|^2 \right) \\
&= e^4 \left(\frac{4u^2}{s^2} + \frac{4t^2}{s^2} + \frac{4t^2}{s^2} + \frac{4u^2}{s^2} \right) \\
&= 8e^4 \, \frac{t^2 + u^2}{s^2},
\end{aligned}
$$

in agreement with our previous result.

Example 2: $e^+e^- \to qg\bar{q}$

As a second example we consider the process $e^+e^- \to qg\bar{q}$. At leading order in perturbation theory there are two Feynman diagrams:



The colour decomposition is trivial – there is only one colour structure:

$$\mathcal{A}(p_1, p_2, p_3, p_4, p_5) = (-e)^2 \, g T^a_{ij} \, A(p_1, p_2, p_3, p_4, p_5).$$

We consider the partial amplitude $A(p_1^{\lambda_1}, p_2^{\lambda_2}, p_3^{\lambda_3}, p_4^{\lambda_4}, p_5^{\lambda_5})$:

- With five external particles we start from $2^5 = 32$ helicity amplitudes.
- Due to helicity conservation, only $2^3 = 8$ helicity amplitudes are non-zero.
- The flip identity on the e^+e^--line reduces the number of helcity amplitudes, which need to be calculated down to $2^2 = 4$.

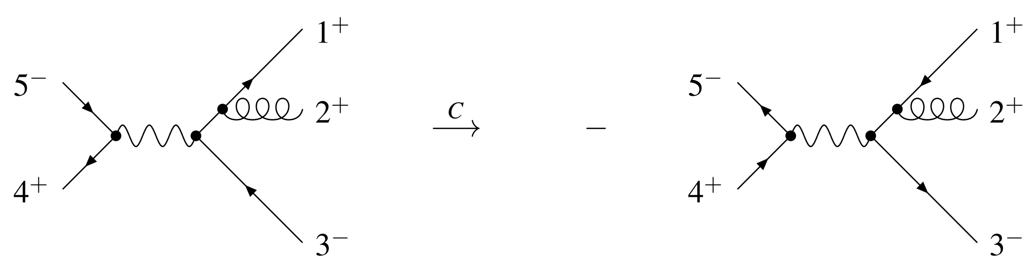
This leaves four helicity amplitudes to be calculated. We may take these four helicity amplitudes to be

$$A(p_1^+, p_2^+, p_3^-, p_4^+, p_5^-), \; A(p_1^+, p_2^-, p_3^-, p_4^+, p_5^-), \; A(p_1^-, p_2^+, p_3^+, p_4^+, p_5^-), \; A(p_1^-, p_2^-, p_3^+, p_4^+, p_5^-).$$

The discrete symmetries of charge conjugation and parity provide additional relations between these helicity amplitudes and reduce the number of helicity amplitudes which need to be calculated to one. Let us first look at charge conjugation:

- exchanges particles $\leftrightarrow$ anti-particles,
- reverses in Feynman diagrams all arrows for fermions,
- multiply by a factor (-1) for each external boson (this is due to the fact that a gauge boson is an eigenstate of the charge conjugation operator with eigenvalue $C = -1$).

Graphically:

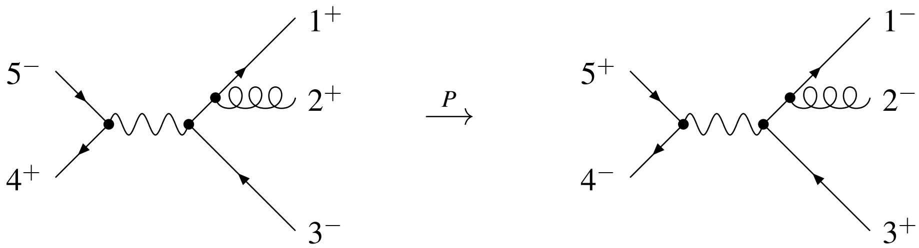


$$A(p_1^+, p_2^+, p_3^-, p_4^+, p_5^-). \quad = \quad -A(p_3^-, p_2^+, p_1^+, p_5^-, p_4^+)$$

Let us now consider parity:

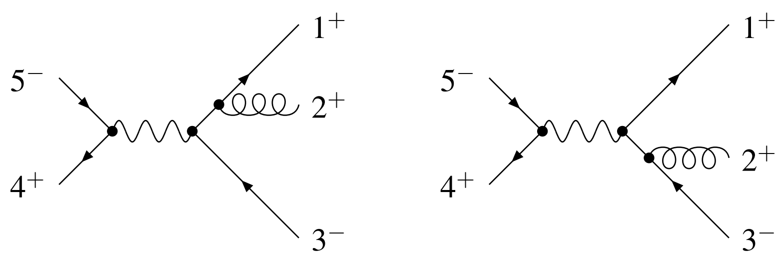
- reverses all external helicities,
- implemented by complex conjugation and multiplication by (-1) for each initial-state fermion (this is due to $p^- < 0$ and $(\sqrt{p^-})^* = -\sqrt{p^-}$ in our definition of the spinors).

Graphically:



$$A(p_1^+,p_2^+,p_3^-,p_4^+,p_5^-) \quad = \quad A(p_1^-,p_2^-,p_3^+,p_4^-,p_5^+)^*.$$

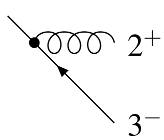
Therefore only one helicity amplitude needs to be calculated, we may take this helicity amplitude to be $A(p_1^+,p_2^+,p_3^-,p_4^+,p_5^-)$. The two Feynman diagrams are



For the polarisation vector of the gluon we use

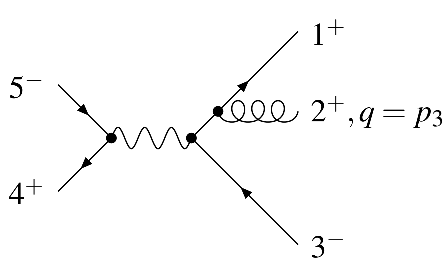
$$\varepsilon_\mu^+(p_2,q) \quad = \quad \frac{\langle q-|\gamma_\mu|\,p_2-\rangle}{\sqrt{2}\,\langle q p_2\rangle}.$$

Let us choose $q = p_3$: Part of second diagram yields



$$...\gamma^\mu\,|3+\rangle\,\frac{\langle 3-|\gamma_\mu|\,2-\rangle}{\sqrt{2}\,\langle 32\rangle} = ...\,|2-\rangle\,\frac{\sqrt{2}}{\langle 32\rangle}\underbrace{\langle 3-|3+\rangle}_{0}.$$

Therefore we need to calculate for this choice of reference momentum only the first diagram:



We have

$$
\begin{aligned}
& A(p_1^+, p_2^+, p_3^-, p_4^+, p_5^-) = \\
= \quad & \left\langle 1+ \left| i\gamma_\mu \frac{i}{\not{1}+\not{2}} i\gamma_\nu \right| 3+ \right\rangle \frac{\left\langle 3- \left|\gamma_\mu\right| 2- \right\rangle}{\sqrt{2}\langle 32 \rangle} \frac{(-i)g^{\nu\rho}}{s_{45}} \left\langle 4+ \left| i\gamma_\rho \right| 5+ \right\rangle \\
= \quad & \frac{(-i)}{\sqrt{2} s_{12} s_{45} \langle 32 \rangle} \left\langle 1+ \left| \gamma_\mu (\not{1}+\not{2}) \gamma_\nu \right| 3+ \right\rangle \langle 3- |\gamma^\mu| 2- \rangle \langle 4+ |\gamma^\nu| 5+ \rangle \\
= \quad & -\sqrt{2} i \frac{1}{s_{12} s_{45} \langle 32 \rangle} [12] \langle 3- | (\not{1}+\not{2}) \gamma_\nu | 3+ \rangle \langle 4+ |\gamma^\nu| 5+ \rangle \\
= \quad & -2\sqrt{2} i \frac{1}{s_{12} s_{45} \langle 32 \rangle} [12] \langle 3- | (\not{1}+\not{2}) | 4- \rangle \langle 53 \rangle .
\end{aligned}
$$

Since $p_1 + p_2 = -p_3 - p_4 - p_5$ we have

$$
\langle 3- | (\not{1}+\not{2}) | 4- \rangle \quad = \quad -\langle 3- |\not{5}| 4- \rangle = -\langle 35 \rangle [54].
$$

This yields

$$
A(p_1^+, p_2^+, p_3^-, p_4^+, p_5^-) \quad = \quad 2\sqrt{2} i \frac{\langle 35 \rangle [54][12] \langle 53 \rangle}{s_{12} s_{45} \langle 32 \rangle}.
$$

With $s_{12} = \langle 12 \rangle [21]$ and $s_{45} = \langle 45 \rangle [54]$ one obtains finally

$$
\begin{aligned}
A(p_1^+, p_2^+, p_3^-, p_4^-, p_5^+) \quad &= \quad 2\sqrt{2} i \frac{\langle 35 \rangle \langle 53 \rangle}{\langle 32 \rangle \langle 21 \rangle \langle 45 \rangle} \\
&= \quad 2\sqrt{2} i \frac{\langle 35 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 54 \rangle}.
\end{aligned}
$$

Example 3: $e^+ e^- \to q g g \bar{q}$

As a final example we consider the process $e^+ e^- \to q g g \bar{q}$. We now have a non-trivial colour decomposition:

$$
\mathcal{A} \quad = \quad (-e)^2 g^2 \left\{ \left(T^a T^b \right)_{ij} A(p_1, p_2, p_3, p_4, p_5, p_6) + \left(T^b T^a \right)_{ij} A(p_1, p_3, p_2, p_4, p_5, p_6) \right\}.
$$

Let us consider the cyclic-ordered partial amplitude $A(p_1, p_2, p_3, p_4, p_5, p_6)$. The Feynman diagrams contributing to this partial amplitude are:

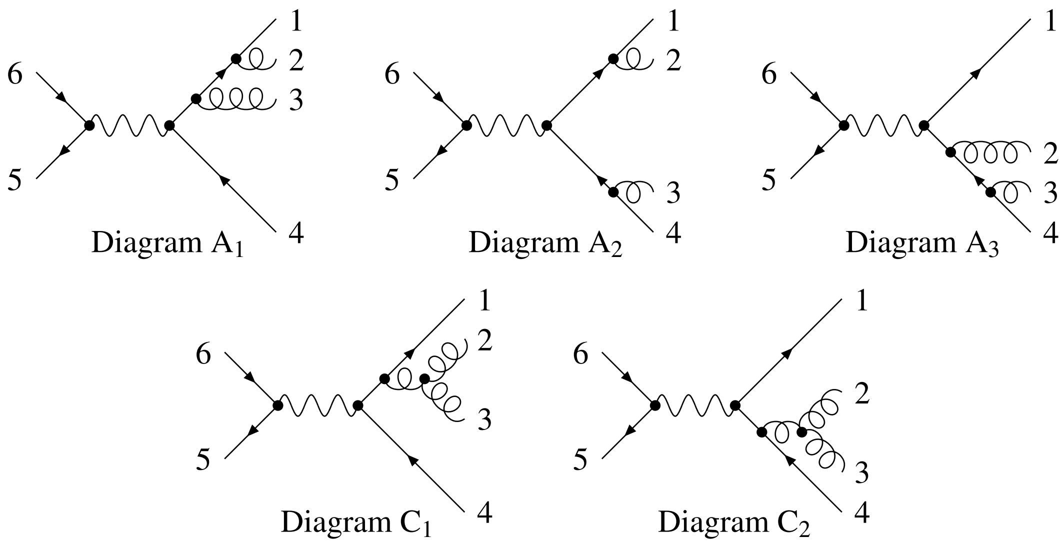


Diagram A_1 Diagram A_2 Diagram A_3

Diagram C_1 Diagram C_2

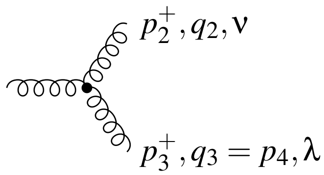
In total there are $2^6 = 64$ helicity amplitudes. Using

- helicity conservation,
- flip identity on the e^+e^--line,
- charge conjugation,
- parity,

it remains to calcualte three distinct helicity amplitudes. These can be taken as

$$A(p_1^+, p_2^+, p_3^+, p_4^-, p_5^+, p_6^-), \quad A(p_1^+, p_2^+, p_3^-, p_4^-, p_5^+, p_6^-), \quad A(p_1^+, p_2^-, p_3^+, p_4^-, p_5^+, p_6^-).$$

As an example we consider here the helicity amplitude $A(p_1^+, p_2^+, p_3^+, p_4^-, p_5^+, p_6^-)$. Choosing as reference momentum $q_3 = p_4$ will eliminate diagrams A_2 and A_3. To choose the reference momentum q_2, we consider the three-gluon vertex, which appears as a building block in diagrams C_1 and C_2.

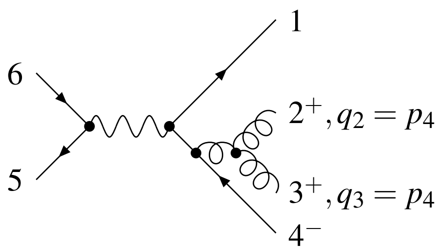


$$= i\left[g^{\nu\lambda}(p_2 - p_3)^\mu + g^{\lambda\mu}(p_2 + 2p_3)^\nu - g^{\mu\nu}(2p_2 + p_3)^\lambda\right] \frac{\langle q_2 - |\gamma_\nu| 2-\rangle}{\sqrt{2}\langle q_2 2\rangle} \frac{\langle 4 - |\gamma_\lambda| 3-\rangle}{\sqrt{2}\langle 43\rangle}$$
$$= \frac{i}{2\langle 2q_2\rangle\langle 34\rangle}\left\{2(p_2 - p_3)^\mu \langle q_2 4\rangle[32] + 2\langle 4 - |\gamma^\mu| 3-\rangle\langle q_2 - |\not{3}| 2-\rangle - 2\langle q_2 - |\gamma^\mu| 2-\rangle\langle 4 - |\not{2}| 3-\rangle\right\}.$$

Choosing $q_2 = p_4$, one obtains for the expression above

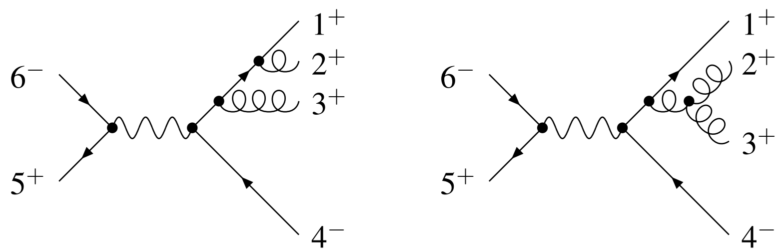
$$\frac{i}{\langle 24\rangle\langle 34\rangle}\left(\langle 4-|\gamma^\mu|3-\rangle\,\langle 4-|\not{3}|2-\rangle - \langle 4-|\gamma^\mu|2-\rangle\,\langle 4-|\not{2}|3-\rangle\right).$$

Plugging this into the diagram C_2, one sees that this diagram vanishes:



$$= \quad \ldots\gamma^\mu|4+\rangle\,\left\langle 4-\left|\gamma_\mu\right|p_j-\right\rangle = 0,$$

where p_j equals p_2 or p_3. Therefore only the diagrams A_1 and C_1 contribute for the choice $q_2 = q_3 = p_4$ to the amplitude $A(p_1^+, p_2^+, p_3^+, p_4^-, p_5^+, p_6^-)$:



Doing the calculation, we find:

$$\begin{aligned}
A \;=\;& \left\langle 1+\left|i\gamma_\mu\frac{i}{\not{1}+\not{2}}i\gamma_\nu\frac{i}{\not{1}+\not{2}+\not{3}}i\gamma_\rho\right|4+\right\rangle \frac{\langle 4-|\gamma^\mu|2-\rangle}{\sqrt{2}\langle 42\rangle}\frac{\langle 4-|\gamma^\nu|3-\rangle}{\sqrt{2}\langle 43\rangle}\frac{(-i)}{s_{56}}\langle 5+|i\gamma^\rho|6+\rangle \\
&+\left\langle 1+\left|i\gamma_\mu\frac{i}{\not{1}+\not{2}+\not{3}}i\gamma_\rho\right|4+\right\rangle\frac{(-i)}{s_{23}}\frac{i}{\langle 24\rangle\langle 34\rangle} \\
&\times\left(\langle 4-|\gamma^\mu|3-\rangle\,\langle 43\rangle[32] - \langle 4-|\gamma^\mu|2-\rangle\,\langle 42\rangle[23]\right)\frac{(-i)}{s_{56}}\langle 5+|i\gamma^\rho|6+\rangle \\
=\;& \frac{2i}{\langle 24\rangle\langle 34\rangle s_{56}}[12]\left\langle 4-\left|\frac{1}{\not{1}+\not{2}}\gamma_\nu\frac{1}{\not{1}+\not{2}+\not{3}}\right|5-\right\rangle\langle 64\rangle\,\langle 4-|\gamma_\nu|3-\rangle \\
&-\frac{4i}{\langle 24\rangle\langle 34\rangle s_{56}}[13]\left\langle 4-\left|\frac{1}{\not{1}+\not{2}+\not{3}}\right|5-\right\rangle\langle 64\rangle\langle 43\rangle[32]\frac{1}{s_{23}} \\
&+\frac{4i}{\langle 24\rangle\langle 34\rangle s_{56}}[12]\left\langle 4-\left|\frac{1}{\not{1}+\not{2}+\not{3}}\right|5-\right\rangle\langle 64\rangle\langle 42\rangle[23]\frac{1}{s_{23}} \\
=\;& \frac{4i\langle 64\rangle}{\langle 24\rangle\langle 34\rangle s_{56}s_{12}s_{23}s_{123}}\left\{s_{23}[12]\,\langle 4-|\not{1}+\not{2}|3-\rangle\,\langle 4-|\not{1}+\not{2}+\not{3}|5-\rangle\right. \\
&-s_{12}[13]\,\langle 4-|\not{1}+\not{2}+\not{3}|5-\rangle\,\langle 43\rangle[32] \\
&\left.+s_{12}[12]\,\langle 4-|\not{1}+\not{2}+\not{3}|5-\rangle\,\langle 42\rangle[23]\right\}.
\end{aligned}$$

With $\langle 4-|\not{1}+\not{2}+\not{3}|5-\rangle = -\langle 4-|\not{6}|5-\rangle = -\langle 46\rangle[65] = \langle 64\rangle[65]$ one obtains:

$$\begin{aligned} A &= \frac{4i\langle 46\rangle^2}{\langle 24\rangle\langle 34\rangle\langle 56\rangle s_{12}s_{23}s_{123}}\left\{[32]\langle 23\rangle[12]\,\langle 4-|\not{1}+\not{2}|3-\rangle - [12]\langle 21\rangle[13]\langle 43\rangle[32]\right. \\ &\quad \left. -s_{12}[12]\langle 42\rangle[32]\right\} \\ &= \frac{4i\langle 46\rangle^2}{\langle 24\rangle\langle 34\rangle\langle 56\rangle\langle 21\rangle\langle 23\rangle s_{123}}\left\{\langle 23\rangle\,\langle 4-|\not{1}+\not{2}|3-\rangle + \langle 12\rangle[13]\langle 43\rangle - s_{12}\langle 42\rangle\right\}. \end{aligned}$$

Since

$$\begin{aligned} \langle 12\rangle\langle 43\rangle[13] &= (\langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle)[13] \\ &= -s_{13}\langle 42\rangle - \langle 23\rangle\langle 41\rangle[13], \end{aligned}$$

we obtain

$$\begin{aligned} A &= \frac{4i\langle 46\rangle^2}{\langle 24\rangle\langle 34\rangle\langle 65\rangle\langle 12\rangle\langle 23\rangle s_{123}}\left(\underbrace{\langle 23\rangle\langle 42\rangle[23] - s_{13}\langle 42\rangle - s_{12}\langle 42\rangle}_{-\langle 42\rangle(s_{12}+s_{23}+s_{13})=\langle 24\rangle s_{123}}\right) \\ &= 4i\frac{\langle 46\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 65\rangle}. \end{aligned}$$

Summary:

$$\begin{aligned} A(q_1^+, \bar{q}_2^-, \bar{e}_3^+, e_4^-) &= 2i\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 43\rangle}, \\ A(q_1^+, g_2^+, \bar{q}_3^-, \bar{e}_4^+, e_5^-) &= 2\sqrt{2}i\frac{\langle 35\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 54\rangle}, \\ A(q_1^+, g_2^+, g_3^+, \bar{q}_4^-, \bar{e}_5^+, e_6^-) &= 4i\frac{\langle 46\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 65\rangle}. \end{aligned}$$