Introduction to Theoretical Elementary Particle Physics:

Relativistic Quantum Field Theory

Part II

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Contents

1 Overview

1.1 Literature

There is no shortage of text books on quantum field theory. I will list a few of them here:

- M. Peskin und D. Schroeder, An Introduction to Quantum Field Theory, Perseus Books, 1995.

- M. Schwartz, Quantum Field Theory and the Standard Model, Cambridge University Press, 2014.

- M. Srednicki, Quantum Field Theory, Cambridge University Press, 2007.

Lecture notes (on more specialised topics):

- S. Weinzierl, The art of computing loop integrals, arXiv:hep-ph/0604068

- S. Weinzierl, Tales of 1001 Gluons, arXiv:1610.05318

2 Review of quantum field theory

We start with a review of the basics of quantum field theory. We will assume that these concepts have been covered in a first course on quantum field theory, therefore the exposition in this section will be brief.

2.1 Path integral formalism

The **generating functional** is given by

$$Z[J(x)] = \mathcal{N} \int \mathcal{D}\phi(x) \, \exp\left[i \int d^4x \, \mathcal{L}(\phi) + J(x)\phi(x) \right].$$

Functional derivatives are defined by

$$\frac{\delta}{\delta J(y)} Z[J(x)] = \lim_{\varepsilon \to 0} \frac{Z[J(x) + \varepsilon \delta(x-y)] - Z[J(x)]}{\varepsilon}.$$

The ***n*-point Green functions** are obtained as functional derivatives of $Z[J]$:

$$\left\langle \Omega \left| T \hat{\phi}(x_1) ... \hat{\phi}(x_n) \right| \Omega \right\rangle = G^n(x_1,...,x_n) = \frac{(-i)^n}{Z[0]} \frac{\delta^n Z[J(x)]}{\delta J(x_1) ... \delta J(x_n)} \Bigg|_{J=0}$$

The functional $Z[J]$ generates all Green functions:

$$Z[J] = Z[0] \sum_n \frac{i^n}{n!} \int d^4x_1 ... d^4x_n \, \left\langle \Omega \left| T \hat{\phi}(x_1) ... \hat{\phi}(x_n) \right| \Omega \right\rangle \, J(x_1) ... J(x_n)$$

For the computation of scattering amplitudes we would like to have as boundary condition not the vacuum but an n particle state. If we assume that interactions are only relevant within a finite volume, we can take this n particle state as the superposition of n non-interacting one-particle states. We call such a state an asymptotic state. If we consider a scalar field theory, the asymptotic field satisfies the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi_{\text{asymp}}(x) \quad = \quad 0.$$

Consider now

$$Z_{\text{asymp}}[J] \quad = \quad \int\limits_{\lim\phi=\phi_{\text{asymp}}} \mathcal{D}\phi \, \exp\left[i\int d^4x \, \mathcal{L}(\phi) + J(x)\phi(x)\right].$$

Then

$$Z_{\text{asymp}}[0] = \sum \frac{i^n}{n!}\int d^4x_1 ... d^4x_n \phi_{\text{asymp}}(x_1)...\phi_{\text{asymp}}(x_n)\left(\Box_{x_1}+m^2\right)...\left(\Box_{x_n}+m^2\right)G^n(x_1,...,x_n).$$

Define now the Fourier transform of the Green functions by

$$G^n(x_1,...,x_n) \quad = \quad \int \frac{d^4p_1}{(2\pi)^4}...\frac{d^4p_n}{(2\pi)^4} e^{-i\sum p_j x_j}(2\pi)^4\,\delta^4\left(p_1+...+p_n\right)\tilde{G}^n(p_1,...,p_n)$$

and the **truncated** (amputated) **Green function in momentum space** by

$$\tilde{G}^n_{\text{trunc}}(p_1,...,p_n) \quad = \quad \left(\frac{i}{p_1^2-m^2}\right)^{-1}...\left(\frac{i}{p_n^2-m^2}\right)^{-1}\tilde{G}^n(p_1,...,p_n).$$

2.2 Cross sections and decay rates

To calculate the cross section at an collider with no initial-state hadrons (e.g. an electron-positron collider):

$$\sigma \quad = \quad \frac{1}{2K(Q^2)}\frac{1}{n_{\text{spin}}(A)n_{\text{spin}}(B)}\int d\phi\left(p_A+p_B;p_1,...,p_n\right)\left|\mathcal{A}\left(p_Ap_B\to p_1p_2...\right)\right|^2$$

where $2K(Q^2)$ is the flux factor and we have $2K(Q^2)=2Q^2$ for massless incoming particles. The scattering amplitudes is given by

$$\mathcal{A}\left(p_Ap_B\to p_1p_2...\right) \quad = \quad \tilde{G}^n_{\text{trunc,connected}}\left(p_Ap_B\to p_1p_2...\right).$$

For a decay rate we have

$$\Gamma \quad = \quad \frac{1}{2m_A}\frac{1}{n_{\text{spin}}(A)}\int d\phi\left(p_A;p_1,...,p_n\right)\left|\mathcal{A}\left(p_A\to p_1p_2...\right)\right|^2.$$

The phase-space measure is given by

$$d\phi(Q;p_1,p_2,...,p_n) \;=\; \frac{1}{\prod_j n_j!}\left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_{\vec{p}_f}}\right)(2\pi)^4\delta^4(Q-\sum p_f),$$

if the final state contains n_j identical particles of type j. If the colliding particles are not elementary (like protons or antiprotons), we have to include the probability of finding the elementary particle A inside the proton or antiproton. If the proton has momentum $\hat{p}_p$ one usually specifies the probability of finding a parton with momentum fraction x by the parton distirbution function

$$f(x).$$

The parton has then the momentum

$$p_A \;=\; xp_p.$$

For the cross section we have to integrate over all possible momentum fractions and the formula for a hadron-hadron collider becomes

$$\begin{aligned}\sigma \;=\; & \int dx_1 f(x_1)\int dx_2 f(x_2)\frac{1}{2K(s)}\frac{1}{n_{\text{spin}}(A)n_{\text{spin}}(B)}\frac{1}{n_{\text{colour}}(A)n_{\text{colour}}(B)} \\ & \int d\phi\,(p_A+p_B;p_1,...,p_n)\,|\mathcal{A}(p_Ap_B\to p_1p_2...)|^2\,.\end{aligned}$$

$n_{\text{colour}}(A)$ and $n_{\text{colour}}(B)$ are the number of colour degrees of the initial state particles.

2.3 Gauge theory

An important example of a quantum field theory is given by gauge theories. Gauge theories (also called Yang-Mills theories) are characterised by the fact that at each point in space-time we have an internal symmetry group G. Let us now briefly review Yang-Mills theory.

Let G be a **Lie group**, $\mathfrak{g}$ its **Lie algebra** and T^a the **generators** of the Lie algebra where the index a takes values from 1 to $\dim G$. We use the conventions

$$\left[T^a,T^b\right] = if^{abc}T^c, \qquad \text{Tr}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab}.$$

The Lie group is called abelian, if all structure constants f^{abc} vanish, otherwise it is called non-abelian. In particle physics we encounter abelian and non-abelian gauge groups. An example for the abelian case is given by quantum electrodynamics (QED), corresponding to the abelian gauge group $U(1)$. An example for the non-abelian case is given by quantum chromodynamics (QCD), corresponding to the non-abelian gauge group $SU(3)$. The extension towards non-abelian gauge groups was first suggested by Yang and Mills in 1954, hence the name "Yang-Mills theory" as a synonym for "gauge theory". We may view the abelian case as a special case of the general non-abelian case.

The **field strength tensor** is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$

where a is an index running over all generatros of the Lie algebra. For a $SU(N)$ gauge group the index a runs from 1 to N^2-1. The Lagrange density reads:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}.$$

The Lagrange density is invariant under the local transformations

$$T^a A^a_\mu(x) \rightarrow U(x) \left(T^a A^a_\mu(x) + \frac{i}{g} \partial_\mu \right) U^\dagger(x)$$

with

$$U(x) = \exp\left(-iT^a \theta_a(x)\right).$$

The action is given by the integral over the Lagrange density:

$$S = \int d^4x \, \mathcal{L}.$$

The quantity

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi} \left(\partial^\mu A^a_\mu\right) \left(\partial^\nu A^a_\nu\right)$$

is called the **gauge-fixing term**, the quantity

$$\mathcal{L}_{\mathrm{FP}} = -\bar{c}^a \partial^\mu D^{ab}_\mu c^b$$

the **Faddeev-Popov term**.

2.4 Fermions in the fundamental representation of the gauge group

We recall that the Lagrange density for a free fermion (e.g. no interactions) is given by

$$\mathcal{L}_F = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

We are now looking for a Lagrange density for the fermionic sector, which remains invariant under gauge transformations. Under a gauge transformation a fermion field $\psi_i(x)$ transforms as

$$\begin{aligned} &\psi_i(x) \rightarrow U_{ij}(x)\psi_j(x), \qquad U_{ij}(x) = \exp\left(-iT^a \theta^a(x)\right), \\ &\bar{\psi}_i(x) \rightarrow \bar{\psi}_j(x) U^\dagger_{ji}(x). \end{aligned}$$

$\theta^a(x)$ depends on the space-time coordinates x. For an infinitessimal gauge transformation we have

$$\psi_i(x) \quad \rightarrow \quad (1 - iT^a\theta^a(x))\,\psi_j(x).$$

We immediately see that a fermion mass term

$$-m\bar{\psi}(x)\psi(x)$$

is invariant under gauge transformations. (Note however that in the standard model the fermion masses are generated through the Yukawa couplings to the Higgs field.) But the term involving derivatives is not gauge invariant:

$$\begin{aligned} i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) &\rightarrow i\bar{\psi}(x)U^\dagger(x)\gamma^\mu\partial_\mu\left(U(x)\psi(x)\right) \\ &= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + \underbrace{i\bar{\psi}(x)\gamma^\mu\left(U^\dagger(x)\partial_\mu U(x)\right)\psi(x)}_{\text{extra}}. \end{aligned}$$

The solution comes in the form of the covariant derivative

$$D_\mu \;=\; \partial_\mu - igT^aA^a_\mu(x),$$

where the gauge field transforms as

$$T^aA^a_\mu(x) \quad \rightarrow \quad U(x)\left(T^aA^a_\mu(x) + \frac{i}{g}\partial_\mu\right)U^\dagger(x).$$

Then the combination

$$\begin{aligned} i\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) &= i\bar{\psi}(x)\gamma^\mu\left(\partial_\mu - igT^aA^a_\mu(x)\right)\psi(x) \\ &\rightarrow i\bar{\psi}(x)U^\dagger(x)\gamma^\mu\left[\partial_\mu - igU(x)\left(T^aA^a_\mu(x) + \frac{i}{g}\partial_\mu\right)U^\dagger(x)\right]U(x)\psi(x) \\ &= i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + i\bar{\psi}(x)\gamma^\mu\left(U^\dagger(x)\partial_\mu U(x)\right)\psi(x) \\ &\quad + g\bar{\psi}(x)\gamma^\mu T^aA^a_\mu(x)\psi(x) + i\bar{\psi}(x)\gamma^\mu\left[\left(\partial_\mu U^\dagger(x)\right)U(x)\right]\psi(x) \end{aligned}$$

is invariant.

This gives us the Lagrange density for the fermion sector:

$$\mathcal{L}_{\text{fermions}} \;=\; \sum_{\text{fermions}} \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m_f\right)\psi(x).$$

2.5 Feynman rules for QED and QCD

From the Lagrange density we may derive the Feynman rules. We summarise here the Feynman rules for QED and QCD.

Propagators:

The propagators for the gauge bosons are in the Feynman gauge ($\xi = 1$).

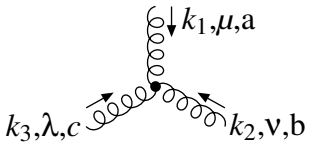
gauge bosons	gluon	A^a_μ	$\frac{-ig_{\mu\nu}}{k^2}\delta_{ab}$
	photon	A_μ	$\frac{-ig_{\mu\nu}}{k^2}$
fermions	quarks	ψ_i	$i\frac{\not{p}+m}{p^2-m^2}\delta_{ij}$
	leptons	ψ	$i\frac{\not{p}+m}{p^2-m^2}$
ghosts		c^a	$\frac{i}{k^2}\delta^{ab}$

Vertices:

Quark-gluon-vertex:

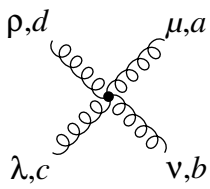
$$ig\gamma_\mu T^a_{ij}$$

3-gluon-vertex:



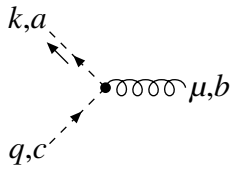
$$gf^{abc}\left[(k_2-k_3)_\mu g_{\nu\lambda}+(k_3-k_1)_\nu g_{\lambda\mu}+(k_1-k_2)_\lambda g_{\mu\nu}\right]$$

4-gluon-vertex:



$$-ig^2\left[f^{abe}f^{ecd}\left(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}\right)+f^{ace}f^{ebd}\left(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\lambda\nu}\right)+f^{ade}f^{ebc}\left(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}\right)\right]$$

Gluon-ghost-vertex:



$$-gf^{abc}k_\mu$$

Fermion-photon-vertex:

$$ieQ\gamma_\mu$$

Additional rules:

An integration

$$\int\frac{d^4k}{(2\pi)^4}$$

for each loop.

A factor (-1) for each closed fermion loop.

Symmetry factor: Multiply the diagram by a factor $1/S$, where S is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed.

External particles:

Outgoing fermion: $\bar{u}(p)$
Outgoing antifermion: $v(p)$

Incoming fermion: $u(p)$
Incoming antifermion: $\bar{v}(p)$

Gauge boson: $\varepsilon_\mu(k)$

Polarisation sums:

$$\begin{aligned}\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) &= \not{p}+m,\\ \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) &= \not{p}-m,\end{aligned}$$

$$\sum_\lambda \varepsilon_\mu^*(k,\lambda)\varepsilon_\nu(k,\lambda) = -g_{\mu\nu}+\frac{k_\mu n_\nu + n_\mu k_\nu}{kn} - n^2\frac{k_\mu k_\nu}{(kn)^2}.$$

Here n^μ is an arbitrary four vector. The dependence on n^μ cancels in gauge-invariant quantities. Using Weyl spinors, a convenient choice of polarisation vectors for the gauge bosons is given by

$$\begin{aligned}\varepsilon_\mu^+(k,q) &= \frac{\langle q-|\gamma_\mu|k-\rangle}{\sqrt{2}\langle qk\rangle},\\ \varepsilon_\mu^-(k,q) &= \frac{\langle q+|\gamma_\mu|k+\rangle}{\sqrt{2}[kq]},\end{aligned}$$

where q^μ is an arbitrary light-like reference momentum. The dependence on q^μ cancels in gauge-invariant quantities.

3 The Standard Model

3.1 Spontaneous symmetry breaking

The concept of gauge theories allowed us to describe successfully quantum electrodynamics and quantum chromodynamcis, the quantum theories of the electromagnetic and the strong force. Both theories are characterised by the fact, that the particles which mediate the forces (photons and gluons) are massless particles. This is required by gauge invariance. In fact, a naive mass term for the gauge bosons in the Lagrangian

$$\mathcal{L}_{\text{mass}} = m^2 A^a_\mu A^{a\,\mu}$$

is not invariant under gauge invariance

$$T^a A^a_\mu(x) \rightarrow U(x)\left(T^a A^a_\mu(x) + \frac{i}{g}\partial_\mu\right) U^\dagger(x).$$

On the other hand it is an experimental fact, that the W-bosons and the Z-boson have non-zero masses. As we do not want to abandon the concept of gauge theories, we face the problem on how to incoporate massive gauge bosons into gauge theories. The solution is provided by the concept of spontaneously broken gauge theories, also known under the name "Higgs mechanism". To start the discussion let us consider a simple physical system with a complex coordinate ϕ and a potential

$$V(\phi) = m^2 |\phi|^2 + \frac{1}{4}\lambda\left(|\phi|^2\right)^2.$$

The potential has a harmonic term $m^2|\phi|^2$ and an anharmonic term $\frac{1}{4}\lambda\left(|\phi|^2\right)^2$. For $m^2 > 0$ and $\lambda > 0$ the potential has an absolut minimum at $\phi = 0$. In classical mechanics the ground state would therefore be $\phi = 0$. This is nothing new.

Imagine now that the potential is given by

$$V(\phi) = -\mu^2 |\phi|^2 + \frac{1}{4}\lambda\left(|\phi|^2\right)^2,$$

with

$$\mu^2 > 0, \qquad \lambda > 0.$$

Then the potential has the shape of a mexican hat, and $\phi = 0$ corresponds to a local maxima. The potential has a minima for

$$|\phi|^2 = \frac{2\mu^2}{\lambda}.$$

The minimas are described by a circle in the complex plane. The ground state of the physical system will be one point of this circle, with no preference for any particular point. Without loss of generality we can choose this point to lie along the positive real axis. Therefore we face the situation, that the potential has a rotational symmetry around the point $\phi = 0$, while the ground state has not. This is the concept of a spontaneously broken symmetry. In general one speaks about a spontaneously broken symmetry, if the Lagrangian of a theory has a certain symmetry, which is not preserved in the ground state of the theory.

3.2 The Higgs mechanism

The standard model is based on the gauge group

$$SU(3) \times SU(2) \times U(1),$$

where $SU(3)$ is the gauge group of the strong interactions, $SU(2)$ the gauge group associated to the weak isospin and $U(1)$ the gauge group associated to the hypercharge. This group is not identical to the gauge group of quantum electrodynamics. To avoid confusions, one often writes $U_Y(1)$ for the group related to the hypercharge and $U_{\mathrm{el-magn}}(1)$ for the gauge group of QED. In the standard model, the electroweak sector $SU(2) \times U_Y(1)$ is spontaneously broken down to $U_{\mathrm{el-magn}}(1)$. We now study the spontaneously symmetry breakdown in detail.

Within the standard model one assumes an additional complex scalar field, transforming as the fundamental representation of $SU(2)$ and having hypercharge $Y = 1$. In the weak isospin space we can write the field as a two-vector with complex entries. It will be conveninet to use the following parametrisation:

$$\phi(x) \quad = \quad \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}.$$

$\phi^+(x)$ is a complex field (two real components), $H(x)$ and $\chi(x)$ are real fields. The quantity v is a real constant. We will later see that it corresponds to the vacuum expectation value of the field $\phi(x)$. The three components $\phi(x)$ and $\chi(x)$ are absorbed as the longitudinal modes of $W_\mu^\pm$ and Z_μ. $H(x)$ is the Higgs field.

The Lagrange density of the Higgs sector

$$\mathcal{L}_{\mathrm{Higgs}} \quad = \quad \left(D_\mu \phi\right)^\dagger (D^\mu \phi) - V(\phi) + \mathcal{L}_{\mathrm{Yukawa}}.$$

The covariant derivative is given by

$$D_\mu \quad = \quad \partial_\mu - igI^a W_\mu^a - ig' \frac{Y}{2} B_\mu,$$

where $I^a = \frac{1}{2}\sigma^a$ (σ^a are the Pauli matrices) and we have $Y = 1$ for the Higgs doublet. (Note that our $g' = -g_1$ (Hollik).)

The Higgs potential is given by

$$V(\phi) = -\mu^2\phi^\dagger\phi + \frac{1}{4}\lambda\left(\phi^\dagger\phi\right)^2.$$

For $\mu^2 > 0$ (and $\lambda > 0$) we have spontaneous symmetry breaking. In that case the potential has a minimum for

$$\phi^\dagger\phi = \frac{2\mu^2}{\lambda} = \frac{v^2}{2}.$$

We have

$$v = 2\sqrt{\frac{\mu^2}{\lambda}}.$$

We write

$$\begin{aligned}\phi(x) &= \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v+H(x)+i\chi(x)) \end{pmatrix}, \\ \phi^\dagger(x) &= \left(\phi^-(x), \frac{1}{\sqrt{2}}(v+H(x)-i\chi(x))\right).\end{aligned}$$

We introduced this parametrisation already previously. We now see that this parametrisation corresponds to an expansion around the minimum of the potential $V(\phi)$. Indeed, we have for $\phi^+(x) = 0$ and $H(x) = \chi(x) = 0$:

$$\phi(x) = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

giving us one point in the minimum of the potential. All points in the minimum of the potential are parametrised with two parameters α and β through

$$\phi(x) = \frac{v}{\sqrt{2}}\begin{pmatrix} e^{i\alpha}\sin\beta \\ e^{i\alpha}\cos\beta \end{pmatrix}.$$

We have to pick a point in the minimum of the potential, any choice of point is as good as any other choice. The choive made above is the conventional choice. With the parametrisation around the minimum as above let us now consider the terms bilinear in the fields W_μ^a and B_μ coming from $(D_\mu\phi)^\dagger(D_\mu\phi)$. We find

$$\begin{aligned}\left(D_\mu\phi\right)^\dagger\left(D_\mu\phi\right)\Big|_{W_\mu^a,B_\mu-\text{bilinear}} &= \frac{1}{8}g^2v^2\left(W_\mu^1W_\mu^1 + W_\mu^2W_\mu^2\right) \\ &\quad + \frac{1}{8}v^2\left(B_\mu, W_\mu^3\right)\begin{pmatrix} g'^2 & -gg' \\ -gg' & g^2 \end{pmatrix}\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}.\end{aligned}$$

We define

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right)$$

and

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}.$$

The angle θ_W is given by

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}.$$

θ_W is called the **Weinberg angle**. We then obtain

$$\begin{aligned}\left(D_\mu\phi\right)^\dagger \left(D_\mu\phi\right)\Big|_{W_\mu^a, B_\mu - \text{bilinear}} &= \frac{1}{2}\left(\frac{vg}{2}\right)^2 \left(W_\mu^{+\,*} W_\mu^+ + W_\mu^{-\,*} W_\mu^-\right) \\ &\quad + \frac{1}{2}\left(\frac{v}{2}\sqrt{g^2+g'^2}\right)^2 (A_\mu, Z_\mu) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}.\end{aligned}$$

We therefore have

$$m_W = \frac{v}{2}g, \qquad m_Z = \frac{v}{2}\sqrt{g^2+g'^2}.$$

It is not too difficult to show that the Higgs mass is given by

$$m_H = \sqrt{\frac{\lambda}{2}}v.$$

In order to see this, we look at all terms bilinear in H from

$$\begin{aligned}-V(\phi)|_{H-\text{bilinear}} &= \mu^2\phi^\dagger\phi - \frac{1}{4}\lambda\left(\phi^\dagger\phi\right)^2\Big|_{H-\text{bilinear}} \\ &= -\frac{1}{2}\left(-\mu^2 + \frac{3}{4}\lambda v^2\right)H^2 = -\frac{1}{2}\left(\frac{\lambda v^2}{2}\right)H^2.\end{aligned}$$

The gauge couplings g and g' are related to the elementary electric charge by

$$e = \frac{gg'}{\sqrt{g^2+g'^2}},$$

or

$$g = \frac{e}{\sin\theta_W}, \qquad g' = \frac{e}{\cos\theta_W}.$$

3.3 Yukawa couplings

The Higgs sector generates also the fermion masses through Yukawa couplings. We discuss this mechanism first in a simplified model without flavour mixing. The full standard model including flavour mixing is discussed in the next section 3.5.

Spin $1/2$ particles are described by four-component spinors $\psi(x)$. With the chiral projectors

$$P_\pm \;=\; \frac{1}{2}(1\pm\gamma_5)$$

we define left- and right-handed spinors:

$$\psi_\pm(x) \;=\; P_\pm\psi(x).$$

The fermions in the standard model can be grouped into three families:

$$\begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} c \\ s \\ \nu_\mu \\ \mu \end{pmatrix}, \quad \begin{pmatrix} t \\ b \\ \nu_\tau \\ \tau \end{pmatrix}.$$

The families differ only by the masses of their members.

Let us now discuss the quantum numbers of the fermions in the electro-weak sector:
The left-handed components (u_L, d_L) and (ν_L, e_L) transform as the fundamental representation under the $SU(2)$ group. The right-handed components u_R, d_R, ν_R and e_R transform as a singlet under the $SU(2)$ group.

In detail one has:

	I_3	Y	Q
u_L	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
d_L	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$
ν_L	$\frac{1}{2}$	-1	0
e_L	$-\frac{1}{2}$	-1	-1

	I_3	Y	Q
u_R	0	$\frac{4}{3}$	$\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
ν_R	0	0	0
e_R	0	-2	-1

The electric charge is given by the **Gell-Mann-Nishijima formula**:

$$Q \;=\; I_3 + \frac{Y}{2}$$

Remark: The table contains a right-handed neutrino, which does not interact with any other particle.

The Yukawa couplings are given by

$$\mathcal{L}_{\text{Yukawa}} = \sum_{\text{families}} \left\{ -\lambda_d \left(\bar{u}_L, \bar{d}_L\right) \phi d_R - \lambda_u \left(\bar{u}_L, \bar{d}_L\right) \phi^C u_R - \lambda_e \left(\bar{\nu}_L, \bar{e}_L\right) \phi e_R - \lambda_\nu \left(\bar{\nu}_L, \bar{e}_L\right) \phi^C \nu_R + \text{ h.c.} \right\},$$

where the charge-conjugate Higgs field is given by

$$\phi^C = 2iI^2\phi^* = i\sigma^2\phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi^-(x) \\ \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \\ -\phi^-(x) \end{pmatrix}.$$

Example:

$$\begin{aligned} -\lambda_d \left(\bar{u}_L, \bar{d}_L\right) \phi d_R + \text{ h.c.} &= -\lambda_d \left(\bar{u}_L, \bar{d}_L\right) \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix} d_R + \text{ h.c.} \\ &= -\frac{v\lambda_d}{\sqrt{2}} \left(\bar{u}_L, \bar{d}_L\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} d_R + \text{ interaction terms} + \text{ h.c.} \\ &= -\frac{v\lambda_d}{\sqrt{2}} \bar{d}_L d_R + \text{ interaction terms} + \text{ h.c.} \end{aligned}$$

Therefore the Yukawa couplings generate the masses of the fermions. From the above example we obtain

$$m_d = \frac{1}{\sqrt{2}} v\lambda_d.$$

The case of the up-type masses is similar. We have for example

$$\begin{aligned} -\lambda_u \left(\bar{u}_L, \bar{d}_L\right) \phi^C u_R + \text{ h.c.} &= -\lambda_u \left(\bar{u}_L, \bar{d}_L\right) \begin{pmatrix} \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x)) \\ -\phi^-(x) \end{pmatrix} u_R + \text{ h.c.} \\ &= -\frac{v\lambda_u}{\sqrt{2}} \left(\bar{u}_L, \bar{d}_L\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_R + \text{ interaction terms} + \text{ h.c.} \\ &= -\frac{v\lambda_u}{\sqrt{2}} \bar{u}_L u_R + \text{ interaction terms} + \text{ h.c.}, \end{aligned}$$

giving us

$$m_u = \frac{1}{\sqrt{2}} v\lambda_u.$$

3.4 Feynman rules in the electroweak sector

In this paragraph we present the most important Feynman rules in the electroweak sector. A complete list of Feynman rules can be found in many textbooks. We present the Feynman rules

for the fields A_μ, Z_μ and $W_\mu^\pm$. The original fields are related to the new fields by

$$\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}, \qquad \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} W_\mu^+ \\ W_\mu^- \end{pmatrix}.$$

As with any gauge theory, we also have to fix the gauge for the electroweak sector. A useful gauge fixing condition is given in the electroweak sector by the 't Hooft gauge (also called R_ξ-gauge):

$$\mathcal{L}_{\rm GF} = \underbrace{-\frac{1}{\xi_W}(\partial^\mu W_\mu^+ - i m_W \xi_W \phi^+)(\partial^\mu W_\mu^- + i m_W \xi_W \phi^-) - \frac{1}{2\xi_Z}(\partial^\mu Z_\mu - m_Z \xi_Z \chi)^2}_{SU(2)} \underbrace{-\frac{1}{2\xi_\gamma}(\partial^\mu A_\mu)^2}_{U(1)}$$

$\xi = 0$ corresponds to Landau gauge, $\xi = 1$ to the Feynman gauge. ϕ^+, ϕ^- and χ are called the would-be Goldstone fields. For the propagators we have to look at all terms in the Lagrangian, which contain exactly two fields. $\mathcal{L}_{\rm Higgs}$ will contain terms, which involve one electroweak gauge boson (Z_μ, $W_\mu^\pm$) and one pseudo-Goldstone field (χ, $\phi^\pm$). These terms lead to a mixing between electroweak gauge bosons and pseudo-Goldstone fields, resulting in a propagator matrix. The 't Hooft R_ξ-gauge eliminates these mixing terms.

The propagators for the W- and Z-bosons are given in 't Hoofts $R_{\xi=1}$-gauge by

gauge bosons	photon	A_μ	$\frac{-ig_{\mu\nu}}{k^2}$
	W-boson	$W_\mu^\pm$	$\frac{-ig_{\mu\nu}}{k^2-m_W^2}$
	Z-boson	Z_μ	$\frac{-ig_{\mu\nu}}{k^2-m_Z^2}$
Higgs sector	Higgs	H	$\frac{i}{k^2-m_H^2}$

Let us now look at the interaction vertices of the electroweak gauge bosons with fermions. We have for the covarariant derivative

$$\begin{aligned} D_\mu &= \partial_\mu - igI^a W_\mu^a - ig'\frac{Y}{2}B_\mu \\ &= \partial_\mu - iQeA_\mu - \frac{ie}{2\sin\theta_W\cos\theta_W}\left(2I^3 - 2Q\sin^2\theta_W\right)Z_\mu \\ &\quad - \frac{ie}{\sqrt{2}\sin\theta_W}\left(I^1 + iI^2\right)W_\mu^+ - \frac{ie}{\sqrt{2}\sin\theta_W}\left(I^1 - iI^2\right)W_\mu^-. \end{aligned}$$

Note that

$$I^1 + iI^2 \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad I^1 - iI^2 \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus we obtain

$$\begin{aligned} \bar{\psi} i\gamma^\mu D_\mu \psi\big|_{\text{trilinear}} \;&=\; \bar{\psi}\gamma^\mu\psi \Big[QeA_\mu + \frac{e}{2\sin\theta_W \cos\theta_W} \left(2I^3 - 2Q\sin^2\theta_W\right) Z_\mu \\ &\quad + \frac{e}{\sqrt{2}\sin\theta_W} \left(I^1 + iI^2\right) W_\mu^+ + \frac{e}{\sqrt{2}\sin\theta_W} \left(I^1 - iI^2\right) W_\mu^- \Big]. \end{aligned}$$

Thus we see that the photon-fermion-antifermion vertex is $iQe\gamma^\mu$, as already known from QED. The Z-fermion-antifermion vertex is given by

$$\frac{ie}{2\sin\theta_W\cos\theta_W}\gamma_\mu\left(v_f - a_f\gamma_5\right),$$

where

$$v_f \;=\; I_3 - 2Q\sin^2\theta_W, \qquad a_f \;=\; I_3,$$

and I_3 equals $1/2$ for up-type fermions and $-1/2$ for down-type fermions. The $\bar{u}_j W^+ d_k$-vertex is given by

$$\frac{ie}{2\sqrt{2}\sin\theta_w}\gamma_\mu(1-\gamma_5)V_{jk},$$

the $\bar{d}_k W^- u_j$-vertex is given by

$$\frac{ie}{2\sqrt{2}\sin\theta_w}\gamma_\mu(1-\gamma_5)V_{jk}^*.$$

In a model without flavour mixing we have $V_{jk} = \delta_{jk}$, in the next paragraph we will study flavour mixing in the Standard model and V_{jk} will be the appropriate matrix element of the quark mixing matrix or the neutrino mixing matrix.

Let us also look at the fermion-antifermion-Higgs vertex. The vertex comes from the Yukawa part of the Lagrange density. As Feynman rules one finds

$$-i\frac{\lambda_f}{\sqrt{2}},$$

where λ_f is the Yukawa coupling of fermion f. Since

$$m_f \;=\; \frac{1}{\sqrt{2}} v\lambda_f \quad \text{and} \quad v \;=\; \frac{2m_W}{e}\sin\theta_W$$

we may equally write for the fermion-antifermion-Higgs vertex

$$-\frac{ie}{2\sin\theta_w}\frac{m_f}{m_W}.$$

3.5 Flavour mixing

We now come to the final ingredient of the standard model: flavour mixing. Let us consider the electroweak sector, in particular the coupling of quarks to the electroweak gauge bosons. Recall the Lagrange density for the quark sector:

$$\mathcal{L}_{\text{fermions}} = \sum_{\text{families}} \left\{ (\bar{u}_L, \bar{d}'_L)\, i\gamma^\mu D_\mu \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{u}_R i\gamma^\mu D_\mu u_R + \bar{d}'_R i\gamma^\mu D_\mu d'_R \right\}$$

For reasons, which will become clear later, we put a prime on all d-type quark fields. The Lagrange density is obtained by replacing the ordinary derivative ∂_μ with the covariant derivative

$$D_\mu = \partial_\mu - igI^a W^a_\mu - ig'\frac{Y}{2}B_\mu.$$

This is required by gauge invariance. This Lagrange density does not allow for mixing between the various quark flavours.

On the other hand the Yukawa couplings are given by

$$\mathcal{L}_{\text{Yukawa}} = \sum_{\text{families}} \left\{ -\lambda_d \left(\bar{u}_L, \bar{d}_L\right) \phi d_R - \lambda_u \left(\bar{u}_L, \bar{d}_L\right) \phi^C u_R + \text{ h.c.} \right\}$$

where the charge-conjugate Higgs field is given by

$$\phi^C = i\sigma_2 \phi^* = 2iI_2\phi^*.$$

Note that now the prime is missing on the d-type quark fields. We have seen that the Yukawa terms lead to mass terms for the fermions:

$$\begin{aligned} &-\lambda_d \left(\bar{u}_L, \bar{d}_L\right) \phi d_R - \lambda_u \left(\bar{u}_L, \bar{d}_L\right) \phi^C u_R + \text{ h.c.} = \\ &\quad -\frac{v\lambda_d}{\sqrt{2}}\bar{d}_L d_R - \frac{v\lambda_u}{\sqrt{2}}\bar{u}_L u_R + \text{ h.c.} + \text{ interaction terms} \end{aligned}$$

However, the Yukawa couplings are not constrained by any gauge symmetry and we could allow for flavour mixing in the Yukawa terms. In fact, nature has chosen this possibility. We therefore consider a general mass term of the form

$$\mathcal{L}_{\text{mass}} = \sum_{\text{families}} \bar{d}''_L M_d d''_R + \bar{u}''_L M_u u''_R + \text{ h.c.}$$

where M_d and M_u are (arbitrary) complex 3×3 matrices in family space. A matrix M can be diagonalised by a biunitary transformation

$$V^{-1} M W = \tilde{M},$$

where $\tilde{M}$ is a diagonal matrix.
Proof: Using the polar decomposition, M can be written as

$$M = HU,$$

where H is hermitian and U is a unitary matrix. H can be diagonalised by a unitary matrix V:

$$V^{-1}HV = \tilde{M},$$

therefore $W = U^{-1}V$.

The gauge part of the Lagrange density

$$\mathcal{L}_{\text{fermions}} = \sum_{\text{families}} \left\{ (\bar{u}_L, \bar{d}'_L)\, i\gamma^\mu D_\mu \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{u}_R i\gamma^\mu D_\mu u_R + \bar{d}'_R i\gamma^\mu D_\mu d'_R \right\}$$

is invariant under the rotations with respect to the family index:

$$\begin{aligned} \begin{pmatrix} u_L \\ d'_L \end{pmatrix} &\to S_L \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ u_R &\to S_{R,u} u_R \\ d_R &\to S_{R,d} d_R \end{aligned}$$

Using this freedom we have with

$$M_u = V_u \tilde{M}_u W_u^{-1}, \qquad M_d = V_d \tilde{M}_d W_d^{-1},$$

$$\sum_{\text{families}} \underbrace{\bar{d}''_L V_d}_{\bar{d}'_L V_u^{-1} V_d} \tilde{M}_d \underbrace{W_d^{-1} d''_R}_{d'_R} + \underbrace{\bar{u}''_L V_u}_{\bar{u}'_L} \tilde{M}_u \underbrace{W_u^{-1} u''_R}_{u'_R} + \text{h.c.} =$$

$$\sum_{\text{families}} \underbrace{\bar{d}'_L V_u^{-1} V_d}_{\bar{d}_L} \tilde{M}_d d'_R + \bar{u}'_L \tilde{M}_u u'_R + \text{ h.c.}$$

$V_u^{-1} V_d$ describes the quark mixing and is a unitary 3×3 matrix:

$$V_{\text{CKM}} = V_u^{-1} V_d.$$

Note that

$$d' = V_{\text{CKM}} d \quad \text{and} \quad \bar{d}' V_{\text{CKM}} = \bar{d}.$$

A unitary $n \times n$ matrix is decribed by n^2 real parameters, out of these

$$\frac{n(n+1)}{2}$$

are phases. For 3×3 matrices we have three angles and six phases. We still have the freedom to redefine our fields by a unitary diagonal matrix:

$$\psi \to \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix} \psi$$

This can be used to eliminate $2n-1$ phases, e.g. five out of the six phases for three generations. This leaves one "physical" phase in the CKM matrix.

Standard parameterisations:

The CKM matrix connects the weak eigenstates (d',s',b') with the mass eigenstates (d,s,b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Standard parametrisation:

$$V_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. The standard parametrisation can be written as a product of three simpler matrices:

$$V_{\rm CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is basically a parametrisation in terms of three Euler angles and one phase.

Wolfenstein parametrisation:

A second parametrisation, the Wolfenstein parametrisation, is quite useful in the quark sector. The usefulness stems from the fact that in the quark sector the CKM matrix is hierarchically ordered. The Wolfenstein parametrisation is an approximate parametrisation, given by

$$V_{\rm CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O\left(\lambda^4\right).$$

Neutrino mixing:

In the lepton sector one uses for Dirac neutrinos the lepton mixing matrix

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

ν'_e, ν'_μ and ν'_τ are the weak eigenstates, whereas ν_1, ν_2 and ν_3 are the mass eigenstates.

GIM mechanism:

The coupling of the Z-boson to the quarks is flavour-neutral, e.g. there are no flavour-changing neutral currents (FCNC).

$$\bar{\psi}'\left(g\cos\theta_W I^3 - g'\sin\theta_W\frac{Y}{2}\right)\gamma_\mu Z^\mu\psi' \;=\; \frac{e}{\sin\theta_W\cos\theta_W}\bar{\psi}'\left(\cos^2\theta_W I^3 - \sin^2\theta_W\frac{Y}{2}\right)\gamma_\mu Z^\mu\psi'.$$

The interaction involves only the diagonal matrices I^3 and $\mathbf{1}$. The primed fields with charge $-1/3$ occur in the form

$$\begin{aligned}
\bar{d}'_L\gamma_\mu Z^\mu d'_L + \bar{s}'_L\gamma_\mu Z^\mu s'_L + \bar{b}'_L\gamma_\mu Z^\mu b'_L &= \left(\bar{d}'_L,\bar{s}'_L,\bar{b}'_L\right)\gamma_\mu Z^\mu\begin{pmatrix} d'_L\\ s'_L\\ b'_L\end{pmatrix}\\
&= \left(\bar{d}_L,\bar{s}_L,\bar{b}_L\right)V^\dagger_{CKM}\gamma_\mu Z^\mu V_{CKM}\begin{pmatrix} d_L\\ s_L\\ b_L\end{pmatrix}\\
&= \left(\bar{d}_L,\bar{s}_L,\bar{b}_L\right)\gamma_\mu Z^\mu\underbrace{V^\dagger_{CKM}V_{CKM}}_{1}\begin{pmatrix} d_L\\ s_L\\ b_L\end{pmatrix}.
\end{aligned}$$

Historical note: Around 1970 only the up-, down- and strange quarks were known. Cabibbo proposed already in 1963 that the linear combination

$$d' \;=\; \cos\theta_C d + \sin\theta_C s$$

enters the weak part of the Lagrangian. This model would predict flavour-changing neutral currents:

$$\bar{d}'\gamma_\mu Z^\mu d' = \underbrace{\cos^2\theta_C\bar{d}\gamma_\mu Z^\mu d + \sin^2\theta_C\bar{s}\gamma_\mu Z^\mu s}_{\Delta S=0} + \underbrace{\sin\theta_C\cos\theta_C\bar{d}\gamma_\mu Z^\mu s + \sin\theta_C\cos\theta_C\bar{s}\gamma_\mu Z^\mu d}_{\Delta S=1}$$

This model is in conflict with experimental observations, for example the ratio of neutral-current ($\Delta S = 1$) to charged-current rates in kaon decay is

$$\frac{K^+\to\pi^+\nu\bar{\nu}}{K^+\to\pi^0\mu^+\nu_e} < 10^{-5}.$$

Glashow, Iliopoulos and Maiani postulated a fourth quark as a isospin partner of s_L and used the mixing matrix between down- strange quarks:

$$\begin{pmatrix} d'\\ s'\end{pmatrix} \;=\; \begin{pmatrix} \cos\theta_C & \sin\theta_C\\ -\sin\theta_C & \cos\theta_C\end{pmatrix}\begin{pmatrix} d\\ s\end{pmatrix}$$

Then

$$\bar{d}'\gamma_\mu Z^\mu d' + \bar{s}'\gamma_\mu Z^\mu s' \;=\; \bar{d}\gamma_\mu Z^\mu d + \bar{s}\gamma_\mu Z^\mu s,$$

an no flavour-changing neutral currents occur at tree level.

3.6 Summary of the Standard Model

Let us summarise the standard model of particle physics. The Lagrange density for the standard model is split into three parts:

$$\mathcal{L}_{\rm SM} \;=\; \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm fermions} + \mathcal{L}_{\rm Higgs}.$$

The Lagrange density for the gauge bosons:

$$\mathcal{L}_{\rm gauge} \;=\; \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}_{SU(3)}\;\underbrace{-\frac{1}{4}W^a_{\mu\nu}W^{\mu\nu a}}_{SU(2)}\;\underbrace{-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}}_{U(1)}\;+\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FP},$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_3 f^{abc}_{SU(3)} A^b_\mu A^c_\nu, \\ W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc}_{SU(2)} W^b_\mu W^c_\nu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned}$$

For $SU(3)$, the indices a, b and c label the generators of $SU(3)$ and run from 1 to 8. For $SU(2)$, they label the generators of $SU(2)$ and run from 1 to 3. The gauge fixing part ('t Hooft gauge):

$$\begin{aligned} \mathcal{L}_{\rm GF} = &\underbrace{-\frac{1}{2\xi_g}(\partial^\mu A^a_\mu)^2}_{SU(3)} \\ &\underbrace{-\frac{1}{\xi_W}(\partial^\mu W^+_\mu - im_W\xi_W\phi^+)(\partial^\mu W^-_\mu + im_W\xi_W\phi^-) - \frac{1}{2\xi_Z}(\partial^\mu Z_\mu - m_Z\xi_Z\chi)^2}_{SU(2)}\;\underbrace{-\frac{1}{2\xi_\gamma}(\partial^\mu A_\mu)^2}_{U(1)} \end{aligned}$$

$\xi = 0$ corresponds to Landau gauge, $\xi = 1$ to the Feynman gauge. ϕ^+, ϕ^- and χ are called the would-be Goldstone fields and have their origin in the Higgs sector. The fields W^a_μ and B_μ are related to the $W^\pm_\mu$, Z_μ and A_μ fields as follows:

$$\begin{aligned} W^\pm_\mu &= \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right), \\ \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}. \end{aligned}$$

The Faddeev-Popov term for QCD reads:

$$\mathcal{L}_{\rm FP} \;=\; \bar{c}^a\left(-\partial^\mu D^{ab}_\mu\right)c^b$$

The covariant derivative in the fundamental representation reads

$$D^{ab}_\mu \;=\; \delta^{ab}\partial_\mu - g_3 f^{abc}_{SU(3)} A^c_\mu.$$

In the electroweak sector we have the ghost fields $d^{\pm}(x)$, $d^{Z}(x)$ and $d^{\gamma}(x)$. The Faddeev-Popov term has the form

$$\mathcal{L}_{\rm FP} \;=\; \bar{d}^{\alpha} K^{\alpha\beta} d^{\beta}.$$

The Lagrange density for the fermion sector:

$$\begin{aligned}\mathcal{L}_{\rm fermions} \;=\; \sum_{\rm families} \Big\{ & \left(\bar{u}_L, \bar{d}'_L\right) i\gamma^{\mu} D_{\mu} \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + \bar{u}_R i\gamma^{\mu} D_{\mu} u_R + \bar{d}'_R i\gamma^{\mu} D_{\mu} d'_R \\ & + \left(\bar{\nu}'_L, \bar{e}_L\right) i\gamma^{\mu} D_{\mu} \begin{pmatrix} \nu'_L \\ e_L \end{pmatrix} + \bar{\nu}'_R i\gamma^{\mu} D_{\mu} \nu'_R + \bar{e}_R i\gamma^{\mu} D_{\mu} e_R \Big\},\end{aligned}$$

$$D_{\mu} \;=\; \begin{cases} \partial_{\mu} - igT^a A^a_{\mu} - igI^a W^a_{\mu} - ig'\frac{Y}{2} B_{\mu}, & \text{quarks,} \\ \partial_{\mu} - igI^a W^a_{\mu} - ig'\frac{Y}{2} B_{\mu}, & \text{leptons.} \end{cases}$$

Note that a right-handed neutrino (with no interactions through D_{μ}) has been added.

The Lagrange density of the Higgs sector

$$\mathcal{L}_{\rm Higgs} \;=\; \left(D_{\mu}\phi\right)^{\dagger} \left(D^{\mu}\phi\right) + \mu^2 \phi^{\dagger}\phi - \frac{1}{4}\lambda \left(\phi^{\dagger}\phi\right)^2 + \mathcal{L}_{\rm Yukawa},$$

The covariant derivative is given as before by

$$D_{\mu} \;=\; \partial_{\mu} - igI^a W^a_{\mu} - ig'\frac{Y}{2} B_{\mu}.$$

The Higgs doublet is parameterised as follows:

$$\begin{aligned}\phi(x) \;&=\; \begin{pmatrix} \phi^{+}(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}, \\ \phi^{\dagger}(x) \;&=\; \left(\phi^{-}(x), \frac{1}{\sqrt{2}}(v + H(x) - i\chi(x))\right).\end{aligned}$$

The Higgs doublet has $Y = 1$.
The Yukawa couplings are given by

$$\mathcal{L}_{\rm Yukawa} \;=\; \sum_{\rm families} \left\{ -\lambda_d \left(\bar{u}_L, \bar{d}_L\right) \phi d_R - \lambda_u \left(\bar{u}_L, \bar{d}_L\right) \phi^C u_R - \lambda_e \left(\bar{\nu}_L, \bar{e}_L\right) \phi e_R - \lambda_{\nu} \left(\bar{\nu}_L, \bar{e}_L\right) \phi^C \nu_R + \text{ h.c.} \right\}$$

The CKM matrix connects the weak eigenstates (d', s', b') with the mass eigenstates (d, s, b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \;=\; \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

In the lepton sector one uses for Dirac neutrinos the lepton mixing matrix

$$\begin{pmatrix} \nu'_e \\ \nu'_{\mu} \\ \nu'_{\tau} \end{pmatrix} \;=\; \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

ν'_e, ν'_{μ} and ν'_{τ} are the weak eigenstates, whereas ν_1, ν_2 and ν_3 are the mass eigenstates.

4 Loop integrals

Up to now we considered only tree-level processes. Let us now turn to quantum corrections. We will now encounter Feynman diagrams with closed loops. Let us look at an example: Fig. 1 shows a Feynman diagram contributing to the one-loop corrections for the process $e^+e^- \to qg\bar{q}$.

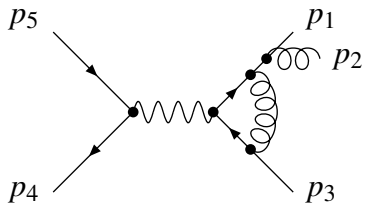


Figure 1: A one-loop Feynman diagram contributing to the process $e^+e^- \to qg\bar{q}$.

At high energies we can ignore the masses of the electron and the light quarks. From the Feynman rules one obtains for this diagram:

$$\begin{aligned}
&\bar{v}(p_4)\,(ie\gamma^\mu)\,u(p_5)\frac{-i}{p_{123}^2} \\
&\quad\times\int\frac{d^4k_1}{(2\pi)^4}\frac{-i}{k_2^2}\bar{u}(p_1)\,(igT^a\not{\varepsilon}(p_2))\,\frac{i\not{p}_{12}}{p_{12}^2}\left(igT^b\gamma_\nu\right)\frac{i\not{k}_1}{k_1^2}\left(ieQ\gamma_\mu\right)\frac{i\not{k}_3}{k_3^2}\left(igT^b\gamma^\nu\right)v(p_3) \\
&= \quad -e^2Qg^3\left(T^aT^bT^b\right)_{ij}\bar{v}(p_4)\gamma^\mu u(p_5)\frac{1}{p_{123}^2}\int\frac{d^4k_1}{(2\pi)^4}\frac{1}{k_2^2}\bar{u}(p_1)\not{\varepsilon}(p_2)\frac{\not{p}_{12}}{p_{12}^2}\gamma_\nu\frac{\not{k}_1}{k_1^2}\gamma_\mu\frac{\not{k}_3}{k_3^2}\gamma^\nu v(p_3).
\end{aligned}$$

Here, $p_{12}=p_1+p_2$, $p_{123}=p_1+p_2+p_3$, $k_2=k_1-p_{12}$, $k_3=k_2-p_3$. Further $\not{\varepsilon}(p_2)=\gamma_\tau\varepsilon^\tau(p_2)$, where $\varepsilon^\tau(p_2)$ is the polarisation vector of the outgoing gluon. All external momenta are assumed to be massless: $p_i^2=0$ for $i=1..5$. We can reorganise this formula into a part, which depends on the loop integration and a part, which does not. The loop integral to be calculated reads:

$$\int\frac{d^4k_1}{(2\pi)^4}\frac{k_1^\rho k_3^\sigma}{k_1^2k_2^2k_3^2},$$

while the remainder, which is independent of the loop integration is given by

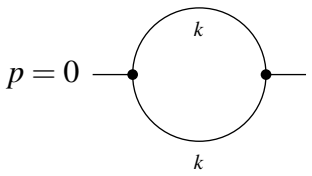
$$-e^2Qg^3\left(T^aT^bT^b\right)_{ij}\bar{v}(p_4)\gamma^\mu u(p_5)\frac{1}{p_{123}^2p_{12}^2}\bar{u}(p_1)\not{\varepsilon}(p_2)\not{p}_{12}\gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma\gamma^\nu v(p_3).$$

The loop integral from the example above contains in the denominator three propagator factors and in the numerator two factors of the loop momentum. We call a loop integral, in which the loop momentum occurs also in the numerator a **tensor integral**. A loop integral, in which the numerator is independent of the loop momentum is called a **scalar integral**. The basic strategy consists in reducing tensor integrals to scalar integrals. The scalar integral associated to our example reads

$$\int\frac{d^4k_1}{(2\pi)^4}\frac{1}{k_1^2k_2^2k_3^2}.$$

4.1 Regularisation

Before we start with the actual calculation of loop integrals, we should mention one complication: Loop integrals are often divergent! Let us first look at the simple example of a scalar two-point one-loop integral with zero external momentum:



$$= \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}$$
$$= \frac{1}{(4\pi)^2}\int_0^\infty dk^2\frac{1}{k^2} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{dx}{x}.$$

This integral diverges at

- $k^2 \to \infty$, which is called an ultraviolet (UV) divergence and at
- $k^2 \to 0$, which is called an infrared (IR) divergence.

Therefore our naive loop integral is ill-defined. The first step to do is to write down a mathematical well-defined expression. To this aim we introduce an (ad-hoc) regularisation scheme. Typical regularisation schemes are:

- Cut-off regularisation:

$$\frac{1}{(4\pi)^2}\int_0^\infty \frac{dx}{x} \;\to\; \frac{1}{(4\pi)^2}\int_\lambda^\Lambda \frac{dx}{x} = \frac{1}{(4\pi)^2}\left[\ln\Lambda - \ln\lambda\right].$$

- Mass regularisation for infrared divergences:

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2} \;\to\; \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)^2} = \frac{1}{(4\pi)^2}\int_0^\infty dk^2\frac{k^2}{(k^2-m^2)^2}.$$

- Dimensional regularisation. Here we first perform the calculation in a space-time of dimension D and continue to $D \to 4$ in the end. It turns out that dimensional regularisation is from a calculational perspective the easiest regularisation scheme. We will treat dimensional regularisation in detail in the next sub-section.

4.2 Loop integration in D dimensions

In this section we will discuss how to perform the D-dimensional loop integrals. It would be more correct to say that we exchange them for some parameter integrals. Our starting point is a one-loop integral with n external legs:

$$\int \frac{d^Dk}{i\pi^{D/2}}\frac{1}{(-P_1)(-P_2)...(-P_n)},$$

where the propagators are of the form

$$P_i = \left(k - \sum_{j=1}^{i} p_j\right)^2 - m_i^2$$

and p_j are the external momenta. The small imaginary parts $i\delta$ are not written explicitly. In the one-loop integral there are some overall factors, which we inserted for convenience: The integral measure is now $d^Dk/(i\pi^{D/2})$ instead of $d^Dk/(2\pi)^D$, and each propagator is multiplied by (-1). The reason for doing this is that the final result will be simpler.

In order to perform the momentum integration we proceed by the following steps:

1. Feynman or Schwinger parametrisation.
2. Shift of the loop momentum to complete the square, such that the integrand depends only on k^2.
3. Wick rotation.
4. Introduction of generalised spherical coordinates.
5. The angular integration is trivial. Using the definitions of Euler's gamma and beta functions, the radial integration can be performed.
6. This leaves only the non-trivial integration over the Feynman parameters.

Although we discuss here only one-loop integrals, the methods presented in this section are rather general and can be applied iteratively to l-loop integrals.

4.2.1 Feynman and Schwinger parameterisation

As already discussed above, the only functions we really want to integrate over D dimensions are the ones which depend on the loop momentum only through k^2. The integrand of the one-loop integral above is not yet in such a form. To bring the integrand into this form, we first convert the product of propagators into a sum. To do this, there are two techniques, one due to Feynman, the other one due to Schwinger. Let us start with the Feynman parameter technique. In its full generality it is also applicable to cases, where each factor in the denominator is raised to some power ν. The formula reads:

$$\prod_{i=1}^{n} \frac{1}{(-P_i)^{\nu_i}} = \frac{\Gamma(\nu)}{\prod_{i=1}^{n}\Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^{n} dx_i\, x_i^{\nu_i-1}\right) \frac{\delta\left(1-\sum_{i=1}^{n} x_i\right)}{\left(-\sum_{i=1}^{n} x_i P_i\right)^{\nu}},$$

$$\nu = \sum_{i=1}^{n} \nu_i.$$

The proof of this formula can be found in many text books and is not repeated here. The price we have to pay for converting the product into a sum are $(n-1)$ additional integrations.

Let us look at a few special cases:

$$\begin{aligned}
\frac{1}{AB} &= \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}, \\
\frac{1}{ABC} &= 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{(xA+yB+(1-x-y)C)^3}, \\
\frac{1}{ABCD} &= 6\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{1}{(xA+yB+zC+(1-x-y-z)D)^4}.
\end{aligned}$$

Let us look at the example from the beginning of this section:

$$\begin{aligned}
\frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} &= 2\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{\left(-x_1k_1^2-x_2k_2^2-x_3k_3^2\right)^3} \\
&= 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\left(-x_1k_1^2-x_2k_2^2-(1-x_1-x_2)k_3^2\right)^3}.
\end{aligned}$$

An alternative to Feynman parameters are Schwinger parameters. Here each propagator is rewritten as

$$\frac{1}{(-P)^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx\, x^{\nu-1} \exp(xP).$$

Therefore we obtain for our example

$$\frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} = \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \, \exp\left(x_1k_1^2+x_2k_2^2+x_3k_3^2\right).$$

4.2.2 Shift of the integration variable

We can now complete the square and shift the loop momentum, such that the integrand becomes a function of k^2. This is best discussed by an example. We consider again the example from above. With $k_2 = k_1 - p_{12}$ and $k_3 = k_2 - p_3$ we have

$$-x_1k_1^2-x_2k_2^2-x_3k_3^2 = -(k_1-x_2p_{12}-x_3p_{123})^2 - x_1x_2s_{12} - x_1x_3s_{123},$$

where $s_{12} = (p_1+p_2)^2$ and $s_{123} = (p_1+p_2+p_3)^2$. We can now define

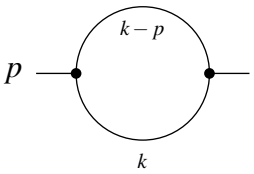
$$k_1' = k_1 - x_2p_{12} - x_3p_{123}$$

and using translational invariance our loop integral becomes

$$\int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} =$$
$$2\int \frac{d^D k_1'}{i\pi^{D/2}} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{\left(-k_1'^{\,2} - x_1 x_2 s_{12} - x_1 x_3 s_{123}\right)^3}.$$

The integrand is now a function of $k_1'^{\,2}$.

Let us look at a second example:



$$\int \frac{d^4 k}{i\pi^2} \frac{1}{k^2 (k-p)^2}$$

Feynman parameterisation leads to

$$\int \frac{d^4 k}{i\pi^2} \frac{1}{k^2 (k-p)^2} = \int_0^1 da \int \frac{d^4 k}{i\pi^2} \frac{1}{\left[ak^2 + (1-a)(k-p)^2\right]^2}.$$

Completing the square we find

$$\begin{aligned} ak^2 + (1-a)(k-p)^2 &= k^2 - 2(1-a)kp + (1-a)p^2 \\ &= \underbrace{[k-(1-a)p]}_{k'}{}^2 + a(1-a)p^2, \end{aligned}$$

and therefore

$$\int \frac{d^4 k}{i\pi^2} \frac{1}{k^2 (k-p)^2} = \int_0^1 da \int \frac{d^4 k'}{i\pi^2} \frac{1}{\left[-k'^2 + a(1-a)(-p)^2\right]^2}.$$

4.2.3 Wick rotation

Having succeeded to rewrite the integrand as a function of k^2, we then perform a Wick rotation, which transforms Minkowski space into an Euclidean space. Remember, that k^2 written out in components in D- dimensional Minkowski space reads

$$k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2 - \ldots$$

(Here k_j denotes the j-th component of the vector k, in contrast to the previous section, where we used the subscript to label different vectors k_j. It should be clear from the context what

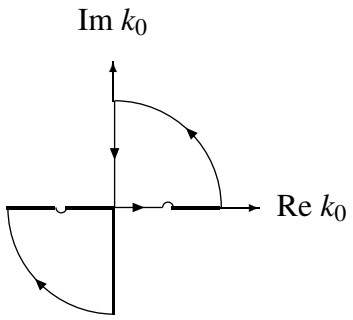


Figure 2: Integration contour for the Wick rotation. The little circles along the real axis exclude the poles.

is meant.) Furthermore, when integrating over k_0, we encounter poles which are avoided by Feynman's $i\delta$-prescription.

$$\frac{i}{k^2 - m^2 + i\delta}.$$

In the complex k_0-plane we consider the integration contour shown in fig. 2. Since the contour does not enclose any poles, the integral along the complete contour is zero:

$$\oint dk_0 f(k_0) \quad = \quad 0.$$

If the quarter-circles at infinity give a vanishing contribution (it can be shown that this is the case) we obtain

$$\int_{-\infty}^{\infty} dk_0 f(k_0) \quad = \quad -\int_{i\infty}^{-i\infty} dk_0 f(k_0).$$

We now make the following change of variables:

$$\begin{aligned} k_0 &= iK_0, \\ k_j &= K_j, \quad \text{for } 1 \leq j \leq D-1. \end{aligned}$$

As a consequence we have

$$\begin{aligned} k^2 &= -K^2, \\ d^D k &= i d^D K, \end{aligned}$$

where K^2 is now given with Euclidean signature:

$$K^2 \quad = \quad K_0^2 + K_1^2 + K_2^2 + K_3^2 + \ldots$$

Combining the exchange of the integration contour with the change of variables we obtain for the integration of a function $f(k^2)$ over D dimensions

$$\int \frac{d^D k}{i\pi^{D/2}} f(-k^2) \;=\; \int \frac{d^D K}{\pi^{D/2}} f(K^2),$$

whenever there are no poles inside the contour of fig. 2 and the arcs at infinity give a vanishing contribution. The integral on the r.h.s. is now over D-dimensional Euclidean space. This equation justifies our conventions, to introduce a factor i in the denominator and a minus sign for each propagator in the defintion of the basic scalar integrals. These conventions are just such that after Wick rotation we have simple formulae.

4.2.4 Generalised spherical coordinates

We now have an integral over D-dimensional Euclidean space, where the integrand depends only on K^2. It is therefore natural to introduce spherical coordinates. In D dimensions they are given by

$$\begin{aligned}
K_0 &= K\cos\theta_1, \\
K_1 &= K\sin\theta_1\cos\theta_2, \\
&\ldots \\
K_{D-2} &= K\sin\theta_1 \ldots \sin\theta_{D-2}\cos\theta_{D-1}, \\
K_{D-1} &= K\sin\theta_1 \ldots \sin\theta_{D-2}\sin\theta_{D-1}.
\end{aligned}$$

In D dimensions we have one radial variable K, $D-2$ polar angles θ_j (with $1 \le j \le D-2$) and one azimuthal angle θ_{D-1}. The measure becomes

$$d^D K \;=\; K^{D-1} dK d\Omega_D,$$

where

$$d\Omega_D \;=\; \prod_{i=1}^{D-1} \sin^{D-1-i}\theta_i \, d\theta_i.$$

Integration over the angles yields

$$\int d\Omega_D \;=\; \int_0^\pi d\theta_1 \sin^{D-2}\theta_1 \ldots \int_0^\pi d\theta_{D-2}\sin\theta_{D-2} \int_0^{2\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)},$$

where $\Gamma(x)$ is Euler's gamma function. Note that the integration on the l.h.s of the equation above is defined for any natural number D, whereas the result on the r.h.s is an analytic function of D, which can be continued to any complex value.

4.2.5 Euler's gamma and beta functions

It is now the appropriate place to introduce two special functions, Euler's gamma and beta functions, which are used within dimensional regularisation to continue the results from integer D towards non-integer values. The gamma function is defined for $\mathrm{Re}(x) > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

It fulfils the functional equation

$$\Gamma(x+1) = x\,\Gamma(x).$$

For positive integers n it takes the values

$$\Gamma(n+1) = n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n.$$

At $x = 1/2$ it has the value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

which can also be inferred from the relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

For integers n we have the reflection identity

$$\frac{\Gamma(x-n)}{\Gamma(x)} = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x+n)}.$$

The gamma function $\Gamma(x)$ has poles located on the negative real axis at $x = 0, -1, -2, \ldots$. Quite often we will need the expansion around these poles. This can be obtained from the expansion around $x = 1$ and the functional equation. The expansion around $\varepsilon = 1$ reads

$$\Gamma(1+\varepsilon) = \exp\left(-\gamma_E \varepsilon + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_n \varepsilon^n\right),$$

where γ_E is Euler's constant

$$\gamma_E = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{j} - \ln n\right) = 0.5772156649...$$

and ζ_n is given by

$$\zeta_n = \sum_{j=1}^{\infty} \frac{1}{j^n}.$$

For example we obtain for the Laurent expansion around $\varepsilon = 0$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon).$$

Euler's beta function is defined for $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$ by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

or equivalently by

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt.$$

The beta function can be expressed in terms of Gamma functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

4.2.6 Result for the momentum integration

We are now in a position to perform the integration over the loop momentum. Let us discuss again the example from the beginning of this section. After Wick rotation we have

$$I = \int \frac{d^Dk_1}{i\pi^{D/2}} \frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} = 2\int \frac{d^DK}{\pi^{D/2}} \int d^3x \frac{\delta(1-x_1-x_2-x_3)}{\left(K^2 - x_1x_2s_{12} - x_1x_3s_{123}\right)^3}.$$

Introducing spherical coordinates and performing the angular integration this becomes

$$I = \frac{2}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dK^2 \int d^3x \frac{\delta(1-x_1-x_2-x_3)\left(K^2\right)^{\frac{D-2}{2}}}{\left(K^2 - x_1x_2s_{12} - x_1x_3s_{123}\right)^3}.$$

For the radial integration we have after the substitution $t = K^2/(-x_1x_2s_{12} - x_1x_3s_{123})$

$$\int_0^\infty dK^2 \frac{\left(K^2\right)^{\frac{D-2}{2}}}{\left(K^2 - x_1x_2s_{12} - x_1x_3s_{123}\right)^3} = \left(-x_1x_2s_{12} - x_1x_3s_{123}\right)^{\frac{D}{2}-3} \int_0^\infty dt \frac{t^{\frac{D-2}{2}}}{(1+t)^3}.$$

The remaining integral is just the second definition of Euler's beta function

$$\int_0^\infty dt \frac{t^{\frac{D-2}{2}}}{(1+t)^3} = \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(3-\frac{D}{2}\right)}{\Gamma(3)}.$$

Putting everything together and setting $D = 4 - 2\varepsilon$ we obtain

$$\int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} = \Gamma(1+\varepsilon) \int d^3x\, \delta(1-x_1-x_2-x_3)\, x_1^{-1-\varepsilon} \left(-x_2 s_{12} - x_3 s_{123}\right)^{-1-\varepsilon}.$$

Therefore we succeeded in performing the integration over the loop momentum k at the expense of introducing a two-fold integral over the Feynman parameters.

As the steps discussed above always occur in any loop integration we can combine them into a master formula. If $\mathcal{U}$ and $\mathcal{F}$ are functions, which are independent of the loop momentum, we have for the integration over Minkowski space with dimension $D = 2m - 2\varepsilon$ (with m being an integer and ε being the dimensional regularisation parameter):

$$\int \frac{d^{2m-2\varepsilon}k}{i\pi^{m-\varepsilon}} \frac{(-k^2)^a}{\left[-\mathcal{U}k^2 + \mathcal{F}\right]^\nu} = \frac{\Gamma(m+a-\varepsilon)}{\Gamma(m-\varepsilon)} \frac{\Gamma(\nu-m-a+\varepsilon)}{\Gamma(\nu)} \frac{\mathcal{U}^{-m-a+\varepsilon}}{\mathcal{F}^{\nu-m-a+\varepsilon}}.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ depend usually on the Feynman parameters and the external momenta and are obtained after Feynman parametrisation from completing the square. They are however independent of the loop momentum k. In the equation above we allowed additional powers $(-k^2)^a$ of the loop momentum in the numerator. This is a slight generalisation and will be useful later. Here we observe that the dependency of the result on a, apart from a factor $\Gamma(m+a-\varepsilon)/\Gamma(m-\varepsilon)$, occurs only in the combination $m+a-\varepsilon = D/2+a$. Therefore adding a power of $(-k^2)$ to the numerator is almost equivalent to consider the integral without this power in dimensions $D+2$.

There is one more generalisation: Sometimes it is convenient to decompose k^2 into a $(2m)$-dimensional piece and a remainder:

$$k_{(D)}^2 = k_{(2m)}^2 + k_{(-2\varepsilon)}^2.$$

If D is an integer greater than $2m$ we have

$$\begin{aligned} k_{(2m)}^2 &= k_0^2 - k_1^2 - \ldots - k_{2m-1}^2, \\ k_{(-2\varepsilon)} &= -k_{2m}^2 - \ldots - k_{D-1}^2. \end{aligned}$$

We also need loop integrals where additional powers of $(-k_{(-2\varepsilon)}^2)$ appear in the numerator. These are related to integrals in higher dimensions as follows:

$$\int \frac{d^{2m-2\varepsilon}k}{i\pi^{m-\varepsilon}} (-k_{(-2\varepsilon)}^2)^r f(k_{(2m)}^\mu, k_{(-2\varepsilon)}^2) = \frac{\Gamma(r-\varepsilon)}{\Gamma(-\varepsilon)} \int \frac{d^{2m+2r-2\varepsilon}k}{i\pi^{m+r-\varepsilon}} f(k_{2m}^\mu, k_{-2\varepsilon}^2).$$

Here, $f(k_{(2m)}^\mu, k_{(-2\varepsilon)}^2)$ is a function which depends on k_{2m}, k_{2m+1}, ..., k_{D-1} only through $k_{(-2\varepsilon)}^2$. The dependency on k_0, k_1, ..., k_{2m-1} is not constrained.
Finally it is worth noting that

$$\int \frac{d^{2m-2\varepsilon}k}{i\pi^{m-\varepsilon}} \left(-k^2\right)^a = \begin{cases} (-1)^a \Gamma(a+1), & \text{if } m+a-\varepsilon = 0, \\ 0, & \text{otherwise.} \end{cases}$$

4.3 Performing the Feynman integrals

Let us summarise what we learned up to now: We had the following cooking recipe for loop integrals: In order to perform the momentum integration we proceed by the following steps:

1. Feynman parametrisation
2. Shift of the loop momentum, such that the denominator has the form $\left(-ck^2 - L\right)^n$.
3. Wick rotation
4. Introduce generalised spherical coordinates
5. The angular integration is trivial. Using the definitions of the gamma- and beta-functions, the radial integration can be performed.
6. This leaves only the non-trivial integration over the Feynman parameters.

We may summarise steps $(3)-(5)$ in a master formula for the integration over the momenta in $D = 2m - 2\varepsilon$ dimensions:

$$\int \frac{d^{2m-2\varepsilon}k}{(2\pi)^{2m-2\varepsilon}i} \frac{(-k^2)^a}{\left[-\mathcal{U}k^2 + \mathcal{F}\right]^n} = \frac{1}{(4\pi)^{m-\varepsilon}} \frac{\Gamma(m+a-\varepsilon)}{\Gamma(m-\varepsilon)} \frac{\Gamma(n-m-a+\varepsilon)}{\Gamma(n)} \frac{\mathcal{U}^{-m-a+\varepsilon}}{\mathcal{F}^{n-m-a+\varepsilon}}$$

Let us now look at a few specific Feynman integrals.

4.3.1 The one-loop tadpole

No Feynman parameterisation needed:

$$\begin{aligned} A_0(m^2) &= -16\pi^2\mu^{2-D} \int \frac{d^Dk}{(2\pi)^D i} \frac{1}{(-k^2+m^2)} \\ &= -16\pi^2\mu^{2\varepsilon} \frac{1}{(4\pi)^{2-\varepsilon}} \Gamma(-1+\varepsilon) \frac{1}{(m^2)^{-1+\varepsilon}} \\ &= -m^2 (4\pi)^{\varepsilon} \Gamma(-1+\varepsilon) \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} \\ &= m^2 \left(\underbrace{\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi}_{\Delta} + 1 - \ln \frac{m^2}{\mu^2} \right). \end{aligned}$$

We recall that γ_E denotes Euler's constant. It will be convenient to define

$$\Delta = \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi.$$

4.3.2 The one-loop two-point function

$$\begin{aligned}
B_0(p^2,m_1^2,m_2^2) &= 16\pi^2\mu^{4-D}\int\frac{d^Dk}{(2\pi)^Di}\,\frac{1}{\left(-k^2+m_1^2\right)\left(-(k-p)^2+m_2^2\right)}\\
&= 16\pi^2\mu^{4-D}\int_0^1 da\int\frac{d^Dk}{(2\pi)^Di}\,\frac{1}{\left[-k^2+a(1-a)(-p^2)+am_1^2+(1-a)m_2^2\right]^2}\\
&= 16\pi^2\mu^{2\varepsilon}\frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\int_0^1 da\left[a(1-a)(-p^2)+am_1^2+(1-a)m_2^2\right]^{-\varepsilon}\\
&= (4\pi)^{\varepsilon}\mu^{2\varepsilon}\Gamma(\varepsilon)\int_0^1 da\left[a(1-a)(-p^2)+am_1^2+(1-a)m_2^2\right]^{-\varepsilon}
\end{aligned}$$

The case $m_1^2=m_2^2=0$:

$$\begin{aligned}
B_0(p^2,0,0) &= (4\pi)^{\varepsilon}\mu^{2\varepsilon}\Gamma(\varepsilon)\int_0^1 da\left[a(1-a)(-p^2)\right]^{-\varepsilon}\\
&= (4\pi)^{\varepsilon}\Gamma(\varepsilon)\left(\frac{-p^2}{\mu^2}\right)^{-\varepsilon}\int_0^1 da\,a^{-\varepsilon}(1-a)^{-\varepsilon}\\
&= (4\pi)^{\varepsilon}\Gamma(\varepsilon)\left(\frac{-p^2}{\mu^2}\right)^{-\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\\
&= \frac{1}{\varepsilon}-\gamma_E+\ln 4\pi+2-\ln\frac{-p^2}{\mu^2}.
\end{aligned}$$

The case $m_1^2=m_2^2=m\neq 0$:

$$\begin{aligned}
B_0(p^2,m^2,m^2) &= (4\pi)^{\varepsilon}\mu^{2\varepsilon}\Gamma(\varepsilon)\int_0^1 da\left[a(1-a)(-p^2)+m^2\right]^{-\varepsilon}\\
&= (4\pi)^{\varepsilon}\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon}\Gamma(\varepsilon)\int_0^1 da\left[1+a(1-a)\left(\frac{-p^2}{m^2}\right)\right]^{-\varepsilon}\\
&= (4\pi)^{\varepsilon}\left(\frac{m^2}{\mu^2}\right)^{-\varepsilon}\Gamma(\varepsilon)2\int_0^{\frac{1}{2}} da\left[1+a(1-a)\left(\frac{-p^2}{m^2}\right)\right]^{-\varepsilon}
\end{aligned}$$

With the substitution

$$b=4a(1-a),\quad a=\frac{1}{2}\left(1-\sqrt{1-b}\right),\quad da=\frac{db}{4\sqrt{1-b}}$$

one obtains

$$\begin{aligned}
B_0(p^2,m^2,m^2) &= (4\pi)^\varepsilon \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} \Gamma(\varepsilon)\frac{1}{2}\int_0^1 db(1-b)^{-\frac{1}{2}}\left[1-bx\right]^{-\varepsilon}, \qquad x=\frac{p^2}{4m^2} \\
&= \frac{1}{2}(4\pi)^\varepsilon \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} \Gamma(\varepsilon)\int_0^1 dbb^{-\frac{1}{2}}\left[1-(1-b)x\right]^{-\varepsilon} \\
&= \frac{1}{2}(4\pi)^\varepsilon \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} \Gamma(\varepsilon)(1-x)^{-\varepsilon}\int_0^1 dbb^{-\frac{1}{2}}\left[1-b\chi\right]^{-\varepsilon}, \quad \chi=\frac{-x}{1-x}=\frac{p^2}{p^2-4m^2}
\end{aligned}$$

Using

$$(1-x)^{-c} = \frac{1}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{\Gamma(n+c)}{\Gamma(n+1)}x^n$$

one obtains

$$\begin{aligned}
B_0(p^2,m^2,m^2) &= \frac{1}{2}(4\pi)^\varepsilon \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} (1-x)^{-\varepsilon}\sum_{n=0}^{\infty}\frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)}\chi^n\int_0^1 db\, b^{n-\frac{1}{2}} \\
&= \frac{1}{2}(4\pi)^\varepsilon \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} (1-x)^{-\varepsilon}\sum_{n=0}^{\infty}\frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)}\frac{\chi^n}{\left(n+\frac{1}{2}\right)}
\end{aligned}$$

Divergent parts can only come from $n=0$, therefore separate the sum into $n=0$ and $n>0$. If we are only interested up to the finite terms, we can set $\varepsilon=0$ in the $n>0$ part.

$$\begin{aligned}
B_0(p^2,m^2,m^2) &= (4\pi)^\varepsilon \left(\frac{m^2}{\mu^2}\right)^{-\varepsilon} \Gamma(\varepsilon)(1-x)^{-\varepsilon}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\chi^n}{n\left(n+\frac{1}{2}\right)} \\
&= \left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi-\ln\frac{m^2}{\mu^2}-\ln(1-x)\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{\left(n+\frac{1}{2}\right)}\right)\chi^n
\end{aligned}$$

We further have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\chi^n}{n} &= -\ln(1-\chi), \\
\sum_{n=1}^{\infty}\frac{\chi^n}{\left(n+\frac{1}{2}\right)} &= \sum_{n=1}^{\infty}\frac{2\sqrt{\chi}^{2n}}{(2n+1)}=\sum_{j=1}^{\infty}\frac{\left(\sqrt{\chi}\right)^j+\left(-\sqrt{\chi}\right)^j}{(j+1)} \\
&= -2+\frac{1}{\sqrt{\chi}}\sum_{j=1}^{\infty}\frac{\left(\sqrt{\chi}\right)^j}{j}-\frac{1}{\sqrt{\chi}}\sum_{j=1}^{\infty}\frac{\left(-\sqrt{\chi}\right)^j}{j}=-2-\frac{1}{\sqrt{\chi}}\ln\left(1-\sqrt{\chi}\right)+\frac{1}{\sqrt{\chi}}\ln\left(1+\sqrt{\chi}\right)
\end{aligned}$$

The case $m_1^2 = m^2 \neq 0, m_2^2 = 0$:

$$\begin{aligned}
B_0(p^2,m^2,0) &= (4\pi)^\varepsilon \mu^{2\varepsilon} \Gamma(\varepsilon) \int\limits_0^1 da\, a^{-\varepsilon} \left[(1-a)(-p^2)+m^2\right]^{-\varepsilon} \\
&= (4\pi)^\varepsilon \left(\frac{m^2-p^2}{\mu^2}\right)^{-\varepsilon} \Gamma(\varepsilon) \int\limits_0^1 da\, a^{-\varepsilon} \left[1-ay\right]^{-\varepsilon}, \quad y = \frac{-p^2}{m^2-p^2} \\
&= (4\pi)^\varepsilon \left(\frac{m^2-p^2}{\mu^2}\right)^{-\varepsilon} \sum_{n=0}^{\infty} \frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)} y^n \int\limits_0^1 da\, a^{n-\varepsilon} \\
&= (4\pi)^\varepsilon \left(\frac{m^2-p^2}{\mu^2}\right)^{-\varepsilon} \sum_{n=0}^{\infty} \frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)} \frac{y^n}{(n+1-\varepsilon)} \\
&= (4\pi)^\varepsilon \left(\frac{m^2-p^2}{\mu^2}\right)^{-\varepsilon} \frac{\Gamma(\varepsilon)}{(1-\varepsilon)} + \sum_{n=1}^{\infty} \frac{y^n}{n(n+1)} \\
&= \left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + 1 - \ln\left(\frac{m^2-p^2}{\mu^2}\right)\right) + 1 + \frac{1-y}{y} \ln(1-y) \\
&= \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + 2 - \ln\left(\frac{m^2-p^2}{\mu^2}\right) - \frac{m^2}{p^2} \ln\left(\frac{m^2}{m^2-p^2}\right)
\end{aligned}$$

4.3.3 More general methods

More complicated integrals are one-loop integrals with more external legs as for example the one-loop three-point function

$$\begin{aligned}
&C_0(p_1^2,p_2^2,p_3^2,m_1^2,m_2^2,m_3^2) = \\
&\quad -16\pi^2\mu^{4-D} \int \frac{d^Dk}{(2\pi)^D i} \frac{1}{\left(-k^2+m_1^2\right)\left(-(k-p_1)^2+m_2^2\right)\left(-(k-p_1-p_2)^2+m_3^2\right)}
\end{aligned}$$

or, in general, integrals with two or even more loops. Methods to tackle these integrals are

- Mellin-Barnes representation

- Nested sums

- Differential equations

- Sector decomposition

4.4 Tensor integrals and Passarino-Veltman reduction

We now consider the reduction of tensor loop integrals (e.g. integrals, where the loop momentum appears in the numerator) to a set of scalar loop integrals (e.g. integrals, where the numerator is

independent of the loop momentum). The loop momentum appears in the numerator for example through the Feynman rules for the quark propagator

$$i\frac{\not{p}+m}{p^2-m^2}$$

or the Feynman rule for the three-gluon vertex

$$gf^{abc}\left[(k_2-k_3)_\mu g_{\nu\lambda}+(k_3-k_1)_\nu g_{\lambda\mu}+(k_1-k_2)_\lambda g_{\mu\nu}\right]$$

For one-loop integrals a systematic algorithm has been first worked out by Passarino and Veltman. The notation for tensor integrals:

$$\begin{aligned}
&B_{0,\mu,\mu\nu}(p,m_1,m_2) = 16\pi^2\mu^{4-D}\int\frac{d^Dk}{(2\pi)^Di}\frac{1,k_\mu,k_\mu k_\nu}{(k^2-m_1^2)((k+p)^2-m_2^2)},\\
&C_{0,\mu,\mu\nu}(p_1,p_2,m_1,m_2,m_3) = 16\pi^2\mu^{4-D}\\
&\quad\int\frac{d^Dk}{(2\pi)^Di}\frac{1,k_\mu,k_\mu k_\nu}{(k^2-m_1^2)((k+p_1)^2-m_2^2)((k+p_1+p_2)^2-m_3^2)}.
\end{aligned}$$

The reduction technique according to Passarino and Veltman consists in writing the tensor integrals in the most general form in terms of form factors times external momenta and/or the metric tensor. For example

$$\begin{aligned}
B^\mu &= p^\mu B_1\\
B^{\mu\nu} &= p^\mu p^\nu B_{21}+g^{\mu\nu}B_{22}
\end{aligned}$$

$$\begin{aligned}
C^\mu &= p_1^\mu C_{11}+p_2^\mu C_{12}\\
C^{\mu\nu} &= p_1^\mu p_1^\nu C_{21}+p_2^\mu p_2^\nu C_{22}+\{p_1p_2\}^{\mu\nu}C_{23}+g^{\mu\nu}C_{24}
\end{aligned}$$

with

$$\{p_1p_2\}^{\mu\nu} = p_1^\mu p_2^\nu+p_1^\nu p_2^\mu$$

One then solves for the form factors B_1, B_{21}, B_{22}, C_{11}, etc. by first contracting both sides with the external momenta and the metric tensor $g^{\mu\nu}$. On the left-hand side the resulting scalar products between the loop momentum k^μ and the external momenta are rewritten in terms of the propagators, as for example

$$2p\cdot k = (k+p)^2-k^2-p^2.$$

The first two terms of the right-hand side above cancel propagators, whereas the last term does not involve the loop momentum anymore. The remaining step is to solve for the formfactors by inverting the matrix which one obtains on the right-hand side of equation (1).

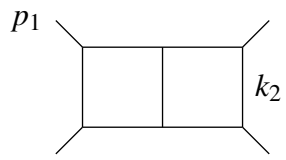


Figure 3: *An example for irreducible scalar products in the numerator: The scalar product $2p_1k_2$ cannot be expressed in terms of inverse propagators.*

Example for the two-point function: Contraction with p_μ or $p_\mu p_\nu$ and $g_{\mu\nu}$ yields

$$p^2 B_1 = \frac{1}{2}\left(A_0(m_1) - A_0(m_2) + \left(m_2^2 - m_1^2 - p^2\right) B_0\right)$$

$$\begin{pmatrix} p^2 & 1 \\ p^2 & D \end{pmatrix} \begin{pmatrix} B_{21} \\ B_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}A_0(m_2) + \frac{1}{2}(m_2^2 - m_1^2 - p^2)B_1 \\ A_0(m_2) + m_1^2 B_0 \end{pmatrix}$$

Solving for the form factors we obtain

$$\begin{aligned} B_1 &= \frac{1}{2p^2}\left(A_0(m_1) - A_0(m_2) + \left(m_2^2 - m_1^2 - p^2\right) B_0\right) \\ B_{21} &= \frac{1}{6p^2}\left(2A_0(m_2) - 2m_1^2 B_0 + 4(m_2^2 - m_1^2 - p^2)B_1 + (\frac{1}{3}p^2 - m_1^2 - m_2^2)\right) \\ B_{22} &= \frac{1}{6}\left(A_0(m_2) + 2m_1^2 B_0 - (m_2^2 - m_1^2 - p^2)B_1 - (\frac{1}{3}p^2 - m_1^2 - m_2^2)\right) \end{aligned}$$

Due to the matrix inversion in the lasr step Gram determinants usually appear in the denominator of the final expression. For a three-point function we would encounter the Gram determinant of the triangle

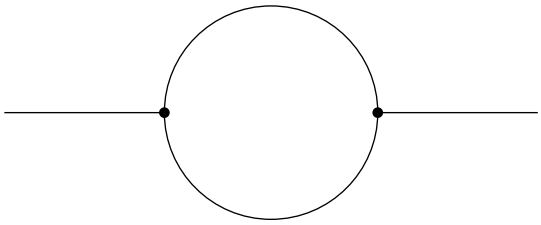
$$\Delta_3 = 4 \begin{vmatrix} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{vmatrix}.$$

One drawback of this algorithm is closely related to these determinants : In a phase space region where p_1 becomes collinear to p_2, the Gram determinant will tend to zero, and the form factors will take large values, with possible large cancellations among them. This makes it difficult to set up a stable numerical program for automated evaluation of tensor loop integrals.

The Passarino-Veltman algorithm is based on the observation, that for one-loop integrals a scalar product of the loop momentum with an external momentum can be expressed as a combination of inverse propagators. This property does no longer hold if one goes to two or more loops. Fig. (3) shows a two-loop diagram, for which the scalar product of a loop momentum with an external momentum cannot be expressed in terms of inverse propagators.

5 Renormalisation

Recall: Loop diagrams are divergent !



$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2} \quad = \quad \frac{1}{(4\pi)^2}\int_0^\infty dk^2 \frac{1}{k^2} = \frac{1}{(4\pi)^2}\int_0^\infty \frac{dx}{x}$$

This integral diverges at

- $k^2 \to \infty$ (UV-divergence) and at
- $k^2 \to 0$ (IR-divergence).

Use dimensional regularisation to regulate UV- and IR-divergences.

Recall the scalar one-loop two-point function for $m_1^2 = m_2^2 = 0$:

$$B_0(p^2,0,0) \quad = \quad \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + 2 - \ln\frac{-p^2}{\mu^2}.$$

Infrared divergences cancel by summing over degenerate states. Ultraviolet divergences are absorbed into a redefinition of the parameters. Example: The renormalisation of the coupling:

$$\underbrace{g_{\text{bare}}}_{\text{divergent}} \quad = \quad \underbrace{Z_g}_{\text{divergent}} \cdot \underbrace{g_{\text{ren}}}_{\text{finite}} \cdot$$

The renormalisation constant Z_g absorbs the divergent part. However Z_g is not unique: One may always shift a finite piece from g_{ren} to Z_g or vice versa. Different choices for Z_g correspond to different renormalisation schemes. Two different renormalisation schemes are always connected by a finite renormalisation. Note that different renormalisation schemes give numerically different answers. Therefore one always has to specify the renormalisation scheme.

Some popular renormalisation schemes:

- On-shell subtraction: Define the renormalisation constants by conditions at a scale where the particles are on-shell $p^2 = m^2$.

- Off-shell subtraction: For massless particles the renormalisation constants in the on-shell scheme would contain an infrared singularity. Therefore, in the off-shell scheme one defines the renormalisation constants by conditions at an unphysical (space-like) scale $p^2 = -\lambda^2$. This scheme is also called momentum-space subtraction scheme.

- Minimal subtraction: The minimal subtraction scheme absorbs exactly the poles in $1/\varepsilon$ into the renormaliztion constants (and nothing else).

- Modified minimal subtraction: As Euler's constant γ_E and $\ln(4\pi)$ always appear in combination with a pole $1/\varepsilon$, the modified minimal subtraction absorbs always the combination

$$\Delta \quad = \quad \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi$$

  into the renormaliztion constants.

5.1 Renormalisation in practice

The modified minimal subtraction is popular in QCD. The Lagrange density of QCD reads

$$\begin{aligned}
\mathcal{L}_{\rm QCD} &= \sum_{\rm quarks} \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \bar{c}^a(x)\left(-\partial^\mu D^{ab}_\mu\right)c^b(x), \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \\
D_\mu &= \partial_\mu - igT^a A^a_\mu, \\
D^{ab}_\mu &= \partial_\mu - g f^{abc} A^c_\mu.
\end{aligned}$$

The Lagrange density depends on the (unrenormalised) fields A^a_μ, ψ, c^a and the (unrenormalised) parameters g, m and ξ. We redefine the fields as follows:

$$A^a_\mu = \sqrt{Z_3}A^a_{\mu,r}, \quad \psi = \sqrt{Z_2}\psi_r, \quad c^a = \sqrt{\tilde{Z}_3}c^a_r.$$

We redefine the parameters as follows:

$$g = Z_g g_r, \quad m = Z_m m_r, \quad \xi = Z_\xi \xi_r = Z_3 \xi_r.$$

Substituting these relations into the Lagrange density we obtain

$$\mathcal{L}_{\rm QCD} \quad = \quad \mathcal{L}_{\rm renorm} + \mathcal{L}_{\rm counterterms},$$

where $\mathcal{L}_{\rm renorm}$ is given by $\mathcal{L}_{\rm QCD}$ where all bare quantities are replaced by renormalised ones. The counterterms are given by

$$\begin{aligned}
\mathcal{L}_{\rm counterterms} &= (Z_2 - 1)\,\bar{\psi}_r\left(i\gamma^\mu\partial_\mu\right)\psi_r - (Z_2 Z_m - 1)\,m_r\bar{\psi}_r\psi_r \\
&\quad + (Z_3 - 1)\,\frac{1}{2}A^a_{\mu,r}\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A^b_{\nu,r} - \left(\tilde{Z}_3 - 1\right)c^a_r\partial^2 c^a_r
\end{aligned}$$

$$
- \left(\underbrace{Z_g Z_3^{\frac{3}{2}}}_{Z_1} - 1 \right) g_r f^{abc} \left(\partial_\mu A^a_{\nu,r} \right) A^b_{\mu,r} A^c_{\nu,r} - \left(\underbrace{Z_g^2 Z_3^2}_{Z_4} - 1 \right) \frac{1}{4} g_r^2 f^{abe} f^{cde} A^a_{\mu,r} A^b_{\nu,r} A^c_{\mu,r} A^d_{\nu,r}
$$

$$
+ \left(\underbrace{Z_g Z_2 \sqrt{Z_3}}_{Z_{1F}} - 1 \right) g_r \bar{\psi}_r T^a \gamma^\mu A^a_{\mu,r} \psi_r - \left(\underbrace{Z_g \tilde{Z}_3 \sqrt{Z_3}}_{\tilde{Z}_1} - 1 \right) g_r f^{abc} \left(\partial^\mu c^a_r \right) c^b_r A^c_{\mu,r}.
$$

The the various constants are not independent, but satisfy

$$
\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} = \frac{Z_4}{Z_1} = Z_g \sqrt{Z_3}.
$$

These are the Slavnov-Taylor-identities (or Ward-Takahashi identities). As a consequence, the coupling constant renormalisation constant may be computed from the corrections to the three-gluon-vertex, the four-gluon-vertex, the quark-gluon-vertex or the ghost-gluon vertex. The Slavnov-Taylor identities guarantees that the result is the same.

5.1.1 Renormalisation of the coupling constant

Let g be the unrenormalised coupling constant, g_r the renormalised coupling constant and g_R the dimensionless renormalised coupling constant. They are related by

$$
\begin{aligned}
g &= Z_g g_r, \\
g_r &= g_R \mu^\varepsilon.
\end{aligned}
$$

From a one-loop calculation one obtains

$$
Z_g = 1 - \frac{1}{6} \left(11 C_A - 4 T_R N_f \right) \frac{g_R^2}{(4\pi)^2} \Delta + O(g_R^4) = 1 - \frac{1}{2} \beta_0 \frac{g_R^2}{(4\pi)^2} \Delta + O(g_R^4),
$$

where as usual

$$
\Delta = \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)
$$

and the coulor factors are

$$
C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_R = \frac{1}{2}.
$$

β_0 is given by

$$
\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_R N_f.
$$

The unrenormalised coupling constant g is of course independent of μ:

$$
\frac{d}{d\mu} g = 0
$$

Therefore

$$\begin{aligned} \mu\frac{d}{d\mu}(Z_g\mu^\varepsilon g_R) &= 0, \\ \mu\frac{d}{d\mu}(Z_g)\mu^\varepsilon g_R+\mu\frac{d}{d\mu}(\mu^\varepsilon)Z_g g_R+\mu\frac{d}{d\mu}(g_R)Z_g\mu^\varepsilon &= 0. \end{aligned}$$

Let us define

$$\beta(g_R) = \mu\frac{d}{d\mu}g_R.$$

Then

$$\begin{aligned} \beta(g_R) &= -\mu^{-\varepsilon}Z_g^{-1}\mu\frac{d}{d\mu}(\mu^\varepsilon)Z_g g_R-\mu^{-\varepsilon}Z_g^{-1}\mu\frac{d}{d\mu}(Z_g)\mu^\varepsilon g_R \\ &= -\varepsilon g_R-\left(Z_g^{-1}\mu\frac{d}{d\mu}Z_g\right)g_R. \end{aligned}$$

Note that the first coefficient of the β-function is calculated as follows:

$$\begin{aligned} Z_g^{-1}\mu\frac{d}{d\mu}Z_g &= Z_g^{-1}\mu\frac{d}{d\mu}\left(1-\frac{1}{2}\beta_0\frac{g_R^2}{(4\pi)^2}\Delta\right) \\ &= Z_g^{-1}\frac{(-\beta_0\Delta)}{(4\pi)^2}g_R\mu\frac{d}{d\mu}g_R \\ &= Z_g^{-1}\frac{(-\beta_0\Delta)}{(4\pi)^2}g_R(-\varepsilon g_R) \\ &= Z_g^{-1}\beta_0\frac{g_R^2}{(4\pi)^2} \\ &= \beta_0\frac{g_R^2}{(4\pi)^2}. \end{aligned}$$

Therefore

$$\beta(g_R) = \mu\frac{d}{d\mu}g_R = -\varepsilon g_R-\beta_0\frac{g_R^3}{(4\pi)^2}.$$

As usual denote

$$\alpha_s = \frac{g_R^2}{4\pi}.$$

We then have

$$\begin{aligned} \mu^2\frac{d}{d\mu^2}\alpha_s &= \frac{1}{2}\mu\frac{d}{d\mu}\alpha_s = \frac{g_R}{4\pi}\mu\frac{d}{d\mu}g_R \\ &= \frac{g_R}{4\pi}\left(-\varepsilon g_R-\beta_0\frac{g_R^3}{(4\pi)^2}\right) \\ &= -\varepsilon\alpha_s-\frac{\beta_0\alpha_s^2}{4\pi}. \end{aligned}$$

Going to $D = 4$ we have therefore

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_s}{4\pi} = -\beta_0 \left(\frac{\alpha_s}{4\pi}\right)^2.$$

At LO the exact solution is given by

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{1}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)},$$

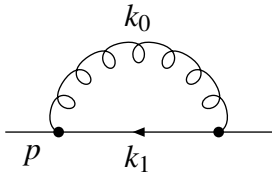
or by

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{\alpha_s(\mu_0)}{4\pi} \frac{1}{1 + \frac{\alpha_s(\mu_0)}{4\pi}\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)},$$

depending on the preferred choice of boundary condition (Λ or $\alpha_s(\mu_0)$).

5.1.2 Mass renormalisation

As an example we consider massive quarks. The quark self-energy is given by the following diagram:



$$\begin{aligned}
-i\Sigma &= g^2 C_F \int \frac{d^D k}{(2\pi)^D} i\gamma_\rho \frac{i}{\not{k}_1 - m} i\gamma^\rho \frac{-i}{k_0^2} \\
&= -i\frac{g^2}{(4\pi)^2} C_F \left\{ (1-\varepsilon)\left[\frac{1}{p^2} A_0(m^2) - \left(1 + \frac{m^2}{p^2}\right) B_0(p^2, m^2, 0)\right] \not{p} + 4m\left(1 - \frac{1}{2}\varepsilon\right) B_0(p^2, m^2, 0) \right\} \\
&= -i\{A\not{p} + Bm\}
\end{aligned}$$

Resummed:

$$\frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma)\frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}\left[(-i\Sigma)\frac{i}{\not{p} - m}\right]^2 = \frac{i}{\not{p} - m}\frac{1}{\left(1 - \frac{\Sigma}{\not{p} - m}\right)} = \frac{i}{\not{p} - m - \Sigma}$$

Modified minimal subtraction

We have for the resummed propagator

$$\begin{aligned}\frac{i}{\not p - m - \Sigma} &= \frac{i}{\not p - m - (A\not p + Bm)} = \frac{i}{(1-A)\not p - (1+B)m} \\ &= \frac{1}{(1-A)}\frac{i}{[\not p - (1+A+B)m]} = \frac{i(1+A)}{\not p - (1+A+B)m}.\end{aligned}$$

Let

$$m = Z_m m_r,$$

where m denotes the unrenormalised mass, Z_m the renormalisation constant and m_r the renormalised mass. We require that

$$Z_m(1+A+B) = \text{finite},$$

therefore

$$Z_m = 1 - (A+B)_{\text{div}}$$

We have

$$\begin{aligned}(A+B)_{\text{div}} &= \frac{g^2}{(4\pi)^2}C_F\left\{\frac{1}{p^2}A_0(m^2) - \left(1+\frac{m^2}{p^2}\right)B_0(p^2,m^2,0) + 4B_0(p^2,m^2,0)\right\}_{\text{div}} \\ &= \frac{g^2}{(4\pi)^2}C_F\left\{\frac{1}{p^2}A_0(m^2) + \left(3-\frac{m^2}{p^2}\right)B_0(p^2,m^2,0)\right\}_{\text{div}} \\ &= \frac{g^2}{(4\pi)^2}C_F\left\{\frac{m^2}{p^2} + \left(3-\frac{m^2}{p^2}\right)\right\}\Delta \\ &= 3\frac{g^2}{(4\pi)^2}C_F\Delta.\end{aligned}$$

Therefore we find that at one-loop-order Z_m is given by

$$Z_m = 1 - 3C_F\frac{g_R^2}{(4\pi)^2}\Delta + O(g_R^4).$$

The anomalous dimension is defined by

$$\gamma^m = \frac{\mu}{Z_m}\frac{dZ_m}{d\mu}$$

Expand γ in powers of $a_s = \alpha_s/(4\pi)$:

$$\begin{aligned}\gamma &= \gamma_0\frac{\alpha_s}{4\pi} + \gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots \\ &= \gamma_0 a_s + \gamma_1 a_s^2 + \ldots\end{aligned}$$

The first coefficients is then given by

$$\gamma_0^m = 6C_F.$$

The running mass:

$$\mu\frac{d}{d\mu}m = 0.$$

Therefore

$$\mu^2\frac{d}{d\mu^2}m_r = -\frac{1}{2}\gamma^m m_r = -\frac{1}{2}\gamma_0^m\frac{\alpha_s}{4\pi}m_r.$$

With

$$\mu^2\frac{d}{d\mu^2} = -\beta_0\left(\frac{\alpha_s}{4\pi}\right)^2\frac{d}{d\frac{\alpha_s}{4\pi}}$$

we find

$$\alpha_s\frac{d}{d\alpha_s}m_r = \frac{\gamma_0}{2\beta_0}m_r.$$

Therefore

$$m_r(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_0}{2\beta_0}} m_r(\mu_0).$$

The on-shell scheme

Recall that we had for the resummed propagator

$$\frac{i}{\not p - m - \Sigma}$$

with

$$\Sigma = A(p^2)\not p + B(p^2)m$$

Note that $A(p^2)$ and $B(p^2)$ are functions of p^2. Expand Σ:

$$\Sigma(\not p) = \Sigma|_{\not p = m_r} + (\not p - m_r)\underbrace{\frac{\partial}{\partial \not p}\Sigma\Big|_{\not p = m_r}}_{\Sigma'} + \ldots$$

Then

$$\begin{aligned}\frac{i}{\not p - m_0 - \Sigma} &= \frac{i}{\not p - m_r - (\not p - m_r)\Sigma' + m_r - m_0 - \Sigma|_{m_r}} \\ &= \frac{i}{(1-\Sigma')(\not p - m_r) + \underbrace{(m_r - m_0 - \Sigma|_{m_r})}_{=0}}.\end{aligned}$$

On-shell condition:

$$m_r - m_0 - \Sigma|_{m_r} = 0,$$

with $m_0 = Z_m m_r$:

$$Z_m = 1 - \frac{1}{m_r}\Sigma|_{m_r}$$

Then

$$\frac{i}{\not p - m_0 - \Sigma} = \frac{i(1+\Sigma')}{\not p - m_r}.$$

Calculation of $\Sigma|_{\not p = m_r}$:

$$\begin{aligned}\Sigma|_{\not p=m_r} &= A(m^2)m + B(m^2)m, \qquad \not p = m + \underbrace{(\not p - m)}_{\text{can be neglected}} \\ &= \frac{g^2}{(4\pi)^2} C_F \left\{ (1-\varepsilon)\left[\frac{1}{m^2}A_0(m^2) - 2B_0(m^2,m^2,0)\right] m + 4m\left(1-\frac{1}{2}\varepsilon\right) B_0(m^2,m^2,0)\right\}\end{aligned}$$

With

$$\begin{aligned}A_0(m^2) &= m^2\left(\Delta + 1 - \ln\frac{m^2}{\mu^2}\right), \\ B_0(m^2,m^2,0) &= \Delta + 2 - \ln\frac{m^2}{\mu^2},\end{aligned}$$

one finds

$$\begin{aligned}\Sigma|_{\not p=m_r} &= \frac{g^2}{(4\pi)^2} C_F \left\{ \left(-\Delta - 2 + \ln\frac{m^2}{\mu^2}\right) m + 4m\left(\Delta + \frac{3}{2} - \ln\frac{m^2}{\mu^2}\right)\right\} \\ &= m\frac{g^2}{(4\pi)^2} C_F \left\{ 3\Delta + 4 - 3\ln\frac{m^2}{\mu^2}\right\}.\end{aligned}$$

Therefore

$$Z_m = 1 - \frac{1}{m_r}\Sigma|_{m_r} = 1 - \frac{\alpha_s}{4\pi} C_F \left(3\Delta + 4 - 3\ln\frac{m^2}{\mu^2}\right).$$

Calculation of Σ':

$$\begin{aligned}\Sigma' &= \frac{\partial}{\partial \not p}\Sigma\bigg|_{\not p=m_r} = \frac{\partial}{\partial \not p}\left(A(p^2)\not p + B(p^2)m\right)\bigg|_{\not p=m_r} \\ &= A(m^2) + \not p \frac{\partial}{\partial \not p} A(p^2)\big|_{\not p=m_r} + m\frac{\partial}{\partial \not p} B(p^2)\big|_{\not p=m_r}\end{aligned}$$

$$
\begin{aligned}
&= A(m^2) + 2\not{p}\not{p}\frac{\partial}{\partial p^2} A(p^2)\big|_{\not{p}=m_r} + 2m\not{p}\frac{\partial}{\partial p^2} B(p^2)\big|_{\not{p}=m_r} \\
&= A(m^2) + 2m^2\frac{\partial}{\partial p^2} A(p^2)\big|_{p^2=m_r^2} + 2m^2\frac{\partial}{\partial p^2} B(p^2)\big|_{p^2=m_r^2} + \ldots, \quad \not{p} = m + \underbrace{(\not{p} - m)}_{\text{can be neglected}} \\
&= A(m^2) + 2m^2\frac{\partial}{\partial p^2}\left(A(p^2) + B(p^2)\right)\big|_{p^2=m_r^2}
\end{aligned}
$$

We keep the Feynman parameter integral, differentiate, do the integral and expand in the end in ε:

$$
\Sigma' = \frac{g^2}{(4\pi)^2} C_F \left(-\frac{3}{\varepsilon} + 3\gamma_E - 3\ln 4\pi - 4 + 3\ln\frac{m^2}{\mu^2} \right)
$$

The propagator is a two-point Green function

$$
G_2(p) \sim \langle 0 | T\psi\psi | 0 \rangle
$$

Under field renormalisation

$$
\psi = \sqrt{Z_2}\psi_r
$$

we have

$$
G_2(p)_{bare} = \left(\sqrt{Z_2}\right)^2 G_2(p)_r
$$

Therefore

$$
G_2(p)_r = \frac{iZ_2^{-1}(1+\Sigma')}{\not{p} - m_r}
$$

The residue of the propagator has to be 1, therefore

$$
Z_2 = 1 + \Sigma' = 1 + \frac{\alpha_s}{4\pi} C_F \left(-3\Delta - 4 + 3\ln\frac{m^2}{\mu^2} \right).
$$

5.2 Renormalisation to all orders

5.2.1 Power counting

The superficial degree of divergence is given by

$$d \cdot l + \sum_v \delta_v - 2n_B - n_F$$

where d is the dimension of space-time, l is the number of loops, δ_v is the number of momentum factors at the vertex v in the diagram G, n_B is the number of internal boson lines and n_F is the number of internal fermion lines.

Power counting theorem: The Feynman integral I_G for the diagram G is absolutely convergent if the superficial degree of divergence is negative for all subdiagrams H of G (including the case $H = G$).

Nested, overlapping and disjoint

If a diagram H_1 is completely included in H_2 as a subdiagram $(H_1 \subseteq H_2)$ we say that H_1 is nested in H_2. If they are not included in each other but have common internal lines and vertices, they are said to overlap. The union $H_1 \cup H_2$ is called the overlapping diagram. If they are neither nested in each other nor overlapping $(H_1 \cap H_2 = \emptyset)$, they are said to be disjoint.

Renormalizability

The BPHZ-method (Bogoliubov and Parasiuk, Acta Math 97, 1957, 227; Hepp, Comm. Math. Phys. 2, 1966, 301; Zimmermann, Comm. Math. Phys. 15, 1969, 208): In dimensional regularizion a Feynman diagram is given as a Laurent series

$$I_G = \sum_{j=-l}^{\infty} a_j \varepsilon^j$$

where l is a positive integer less or equal the number of loops in G. Define an operator T_G for a diagram, which picks out the divergent part, by

$$T_G I_G = \sum_{j=-l}^{-1} a_j \varepsilon^j$$

Define Bogoliubov's R-operation by

$$R_G = (1 - T_G) \prod_{H \in \Phi} (1 - T_H)$$

where Φ is a set of all one-particle irreducible subdiagrams H of G, which are superficially divergent. A one-particle irreducible (or proper) subdiagram H of G, which is superficially divergent

is also called a renormalisation part. If $H_1 \subset H_2$ the order is such that $1 - T_{H_1}$ comes to the right of $1 - T_{H_2}$.

Lemma on overlapping divergences: If two renormalisation parts H_1 and H_2 of G overlap with each other, we have

$$(1 - T_{H_{12}})\, T_{H_1} T_{H_2} \;=\; 0$$

for all suitable renormalisation parts H_{12} of G which includes both H_1 and H_2 as subdiagrams.

5.2.2 Hopf algebras

Let R be a commutative ring with unit 1. An algebra over the ring R is a R-module together with a multiplication $\cdot$ and a unit e. We will always assume that the multiplication is associative. In physics, the ring R will almost always be a field K (examples are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ or the complex number $\mathbb{C}$). In this case the R-module will actually be a K-vector space. Note that the unit can be viewed as a map from R to A and that the multiplication can be viewed as a map from the tensor product $A \otimes A$ to A (e.g. one takes two elements from A, multiplies them and gets one element out).

A coalgebra has instead of multiplication and unit the dual structures: a comultiplication Δ and a counit $\bar{e}$. The counit is a map from A to R, whereas comultiplication is a map from A to $A \otimes A$. Note that comultiplication and counit go in the reverse direction compared to multiplication and unit. We will always assume that the comultiplication is coassociative. The general form of the coproduct is

$$\Delta(a) \;=\; \sum_i a_i^{(1)} \otimes a_i^{(2)},$$

where $a_i^{(1)}$ denotes an element of A appearing in the first slot of $A \otimes A$ and $a_i^{(2)}$ correspondingly denotes an element of A appearing in the second slot. Sweedler's notation consists in dropping the dummy index i and the summation symbol:

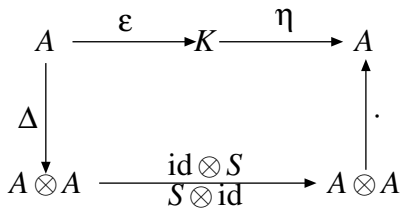
$$\Delta(a) \;=\; a^{(1)} \otimes a^{(2)}$$

The sum is implicitly understood. This is similar to Einstein's summation convention, except that the dummy summation index i is also dropped. The superscripts $^{(1)}$ and $^{(2)}$ indicate that a sum is involved.

A bialgebra is an algebra and a coalgebra at the same time, such that the two structures are compatible with each other. Using Sweedler's notation, the compatibility between the multiplication and comultiplication is expressed as

$$\Delta(a \cdot b) \;=\; \left(a^{(1)} \cdot b^{(1)}\right) \otimes \left(a^{(2)} \cdot b^{(2)}\right).$$

A Hopf algebra is a bialgebra with an additional map from A to A, called the antipode $\mathcal{S}$, which fulfills



$$a^{(1)} \cdot \mathcal{S}\left(a^{(2)}\right) = \mathcal{S}\left(a^{(1)}\right) \cdot a^{(2)} = 0 \qquad \text{for } a \neq e.$$

Examples of Hopf algebras

The group algebra

Let G be a group and denote by KG the vector space with basis G. KG is an algebra with the multiplication given by the group multiplication. The counit $\bar{e}$ is given by:

$$\bar{e}(g) \;=\; 1.$$

The coproduct Δ is given by:

$$\Delta(g) \;=\; g \otimes g.$$

The antipode $\mathcal{S}$ is given by:

$$\mathcal{S}(g) \;=\; g^{-1}.$$

KG is a cocommutative Hopf algebra. KG is commutative if G is commutative.

Lie algebras

A Lie algebra $\mathfrak{g}$ is not necessarily associative nor does it have a unit. To overcome this obstacle one considers the universal enveloping algebra $U(\mathfrak{g})$, obtained from the tensor algebra $T(\mathfrak{g})$ by factoring out the ideal

$$X \otimes Y - Y \otimes X - [X,Y],$$

with $X, Y \in \mathfrak{g}$. The counit $\bar{e}$ is given by:

$$\bar{e}(e) = 1, \qquad \bar{e}(X) = 0.$$

The coproduct Δ is given by:

$$\Delta(e) = e \otimes e, \qquad \Delta(X) = X \otimes e + e \otimes X.$$

The antipode $\mathcal{S}$ is given by:

$$\mathcal{S}(e) = e, \qquad \mathcal{S}(X) = -X.$$

Quantum SU(2)

The Lie algebra $su(2)$ is generated by three generators $H, X_\pm$ with

$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = H.$$

To obtain the deformed algebra $U_q(su(2))$, the last relation is replaced with

$$[X_+, X_-] \;=\; \frac{q^H - q^{-H}}{q - q^{-1}}.$$

The undeformed Lie algebra $su(2)$ is recovered in the limit $q \to 1$. The counit $\bar{e}$ is given by:

$$\bar{e}(e) = 1, \qquad \bar{e}(H) = \bar{e}(X_\pm) = 0.$$

The coproduct Δ is given by:

$$\begin{aligned} \Delta(H) &= H \otimes e + e \otimes H, \\ \Delta(X_\pm) &= X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm. \end{aligned}$$

The antipode $\mathcal{S}$ is given by:

$$\mathcal{S}(H) = -H, \qquad \mathcal{S}(X_\pm) = -q^{\pm 1} X_\pm.$$

Shuffle algebras

Consider a set of letters A. A word is an ordered sequence of letters:

$$w \;=\; l_1 l_2 ... l_k.$$

The word of length zero is denoted by e. A shuffle algebra $\mathcal{A}$ on the vector space of words is defined by

$$(l_1 l_2 ... l_k) \cdot (l_{k+1} ... l_r) \;=\; \sum_{\text{shuffles } \sigma} l_{\sigma(1)} l_{\sigma(2)} ... l_{\sigma(r)},$$

where the sum runs over all permutations σ, which preserve the relative order of $1, 2, ..., k$ and of $k+1, ..., r$. The counit $\bar{e}$ is given by:

$$\bar{e}(e) = 1, \qquad \bar{e}(l_1 l_2 ... l_n) = 0.$$

The coproduct Δ is given by:

$$\Delta(l_1 l_2 ... l_k) \;=\; \sum_{j=0}^{k} \left(l_{j+1} ... l_k\right) \otimes \left(l_1 ... l_j\right).$$

The antipode $\mathcal{S}$ is given by:

$$\mathcal{S}(l_1 l_2 ... l_k) \;=\; (-1)^k \, l_k l_{k-1} ... l_2 l_1.$$

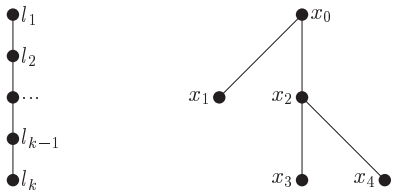


Figure 4: An element of the shuffle algebra can be represented by a rooted tree without side-branchings, as shown in the left figure. The right figure shows a general rooted tree with side-branchings. The root is drawn at the top.

Rooted trees

Consider a set of rooted trees (fig. 4). An admissible cut of a rooted tree is any assignment of cuts such that any path from any vertex of the tree to the root has at most one cut. An admissible cut maps a tree t to a monomial in trees $t_1 \times ... \times t_{n+1}$. Precisely one of these subtrees t_j will contain the root of t. We denote this distinguished tree by $R^C(t)$, and the monomial delivered by the n other factors by $P^C(t)$. The counit $\bar{e}$ is given by:

$$\bar{e}(e) = 1, \qquad \bar{e}(t) = 0 \text{ for } t \neq e.$$

The coproduct Δ is given by:

$$\begin{aligned} \Delta(e) &= e \otimes e, \\ \Delta(t) &= t \otimes e + e \otimes t + \sum_{\text{adm. cuts } C \text{ of } t} P^C(t) \otimes R^C(t). \end{aligned}$$

The antipode $\mathcal{S}$ is given by:

$$\begin{aligned} \mathcal{S}(e) &= e, \\ \mathcal{S}(t) &= -t - \sum_{\text{adm. cuts } C \text{ of } t} \mathcal{S}\left(P^C(t)\right) \times R^C(t). \end{aligned}$$

5.2.3 Renormalisation revisited

Short-distance singularities of the perturbative expansion of quantum field theories require renormalisation. The combinatorics involved in the renormalisation is governed by a Hopf algebra. The model for this Hopf algebra is the Hopf algebra of rooted trees (fig. 5 and 6).

Recall the recursive definition of the antipode:

$$\mathcal{S}(t) = -t - \sum_{\text{adm. cuts } C \text{ of } t} \mathcal{S}\left(P^C(t)\right) \times R^C(t).$$

The antipode satisfies

$$m[(\mathcal{S} \otimes \mathrm{id})\Delta(t)] = 0,$$

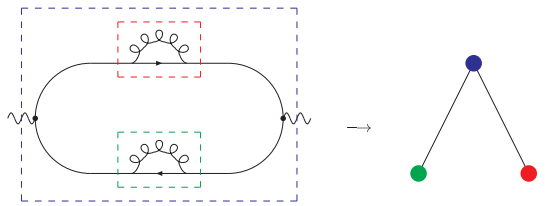


Figure 5: Nested singularities are encoded in rooted trees.

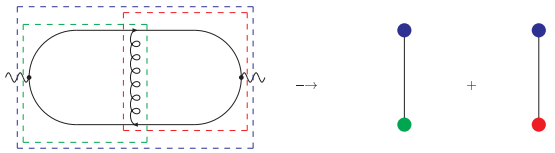


Figure 6: Overlapping singularities yield a sum of rooted trees.

where m denotes multiplication:

$$m(a \otimes b) \quad = \quad a \cdot b.$$

Let $\mathcal{R}$ be an operation which approximates a tree by another tree with the same singularity structure and which satisfies the Rota-Baxter relation:

$$\mathcal{R}(t_1 t_2) + \mathcal{R}(t_1)\,\mathcal{R}(t_2) \quad = \quad \mathcal{R}(t_1 \mathcal{R}(t_2)) + \mathcal{R}(\mathcal{R}(t_1)\, t_2)\,.$$

For example, minimal subtraction ($\overline{MS}$)

$$\mathcal{R}\left(\sum_{k=-L}^{\infty} c_k \varepsilon^k\right) \quad = \quad \sum_{k=-L}^{-1} c_k \varepsilon^k$$

fulfills the Rota-Baxter relation. To simplify the notation, I drop the distinction between a Feynman graph and the evaluation of the graph. One can now twist the antipode with $\mathcal{R}$ and define a new map

$$S_{\mathcal{R}}(t) \quad = \quad -\mathcal{R}\left(t + \sum_{\text{adm. cuts } C \text{ of } t} S_{\mathcal{R}}\left(P^C(t)\right) \times R^C(t)\right).$$

From the multiplicativity constraint it follows that

$$S_{\mathcal{R}}(t_1 t_2) \quad = \quad S_{\mathcal{R}}(t_1)\, S_{\mathcal{R}}(t_2)\,.$$

If we replace S by $S_{\mathcal{R}}$ we obtain

$$m\left[\left(S_{\mathcal{R}} \otimes \mathrm{id}\right) \Delta(t)\right] \quad = \quad \text{finite},$$

since by definition $\mathcal{S}_{\mathcal{R}}$ differs from $\mathcal{S}$ only by finite terms. The formula above is equivalent to the forest formula. It should be noted that $\mathcal{R}$ is not unique and different choices for $\mathcal{R}$ correspond to different renormalisation prescription.

6 Mathematical structures of loop integrals

Recall: Shuffle algebras Consider a set of letters A. A word is an ordered sequence of letters:

$$w = l_1 l_2 ... l_k.$$

The word of length zero is denoted by e. A shuffle algebra $\mathcal{A}$ on the vector space of words is defined by

$$(l_1 l_2 ... l_k) \cdot (l_{k+1} ... l_r) = \sum_{\text{shuffles } \sigma} l_{\sigma(1)} l_{\sigma(2)} ... l_{\sigma(r)},$$

where the sum runs over all permutations σ, which preserve the relative order of $1, 2, ..., k$ and of $k+1, ..., r$. The counit $\bar{e}$ is given by:

$$\bar{e}(e) = 1, \qquad \bar{e}(l_1 l_2 ... l_n) = 0.$$

The coproduct Δ is given by:

$$\Delta(l_1 l_2 ... l_k) = \sum_{j=0}^{k} \left(l_{j+1} ... l_k\right) \otimes \left(l_1 ... l_j\right).$$

The antipode $\mathcal{S}$ is given by:

$$\mathcal{S}(l_1 l_2 ... l_k) = (-1)^k \, l_k l_{k-1} ... l_2 l_1.$$

Recall: Feynman- and Schwinger parameterisation.

$$\prod_{i=1}^{n} \frac{1}{(-P_i)^{\nu_i}} = \frac{\Gamma(\nu)}{\prod_{i=1}^{n} \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^{n} dx_i \, x_i^{\nu_i - 1} \right) \frac{\delta\left(1 - \sum_{i=1}^{n} x_i\right)}{\left(-\sum_{i=1}^{n} x_i P_i\right)^{\nu}},$$

$$\nu = \sum_{i=1}^{n} \nu_i.$$

$$\frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} = 2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)}{\left(-x_1 k_1^2 - x_2 k_2^2 - x_3 k_3^2\right)^3}$$

$$= 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\left(-x_1 k_1^2 - x_2 k_2^2 - (1 - x_1 - x_2) k_3^2\right)^3}.$$

An alternative to Feynman parameters are Schwinger parameters. Here each propagator is rewritten as

$$\frac{1}{(-P)^{\nu}} = \frac{1}{\Gamma(\nu)} \int_0^{\infty} dx \, x^{\nu - 1} \exp(xP).$$

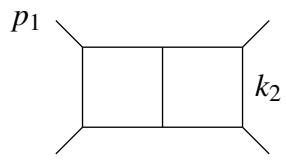


Figure 7: *An example for irreducible scalar products in the numerator: The scalar product $2p_1k_2$ cannot be expressed in terms of inverse propagators.*

Therefore we obtain for our example

$$\frac{1}{(-k_1^2)(-k_2^2)(-k_3^2)} = \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \, \exp\left(x_1k_1^2 + x_2k_2^2 + x_3k_3^2\right).$$

Recall: Tensor reduction at one-loop with Passarino-Veltman algorithm:

$$2p\cdot k = (k+p)^2 - k^2 - p^2.$$

The Passarino-Veltman algorithm is based on the observation, that for one-loop integrals a scalar product of the loop momentum with an external momentum can be expressed as a combination of inverse propagators. This property does no longer hold if one goes to two or more loops. Fig. (7) shows a two-loop diagram, for which the scalar product of a loop momentum with an external momentum cannot be expressed in terms of inverse propagators.

Recall: If $\mathcal{U}$ and $\mathcal{F}$ are functions, which are independent of the loop momentum, we have for the integration over Minkowski space with dimension $D = 2m - 2\varepsilon$:

$$\int \frac{d^{2m-2\varepsilon}k}{i\pi^{m-\varepsilon}} \frac{(-k^2)^a}{\left[-\mathcal{U}k^2 + \mathcal{F}\right]^\nu} = \frac{\Gamma(m+a-\varepsilon)}{\Gamma(m-\varepsilon)} \frac{\Gamma(\nu-m-a+\varepsilon)}{\Gamma(\nu)} \frac{\mathcal{U}^{-m-a+\varepsilon}}{\mathcal{F}^{\nu-m-a+\varepsilon}}.$$

6.1 General tensor integrals

Let us now consider a tensor integral. After the change of variables for the diagonalisation of the quadratic form, we have a polynomial in the Feynman or Schwinger parameters and the loop momentum k in the numerator. Integrals with an odd power of a loop momentum in the numerator vanish by symmetry, while integrals with an even power of the loop momentum can be related by Lorentz invariance to scalar integrals:

$$\begin{aligned}
\int \frac{d^Dk}{i\pi^{D/2}} k^\mu k^\nu f(k^2) &= \frac{1}{D} g^{\mu\nu} \int \frac{d^Dk}{i\pi^{D/2}} k^2 f(k^2), \\
\int \frac{d^Dk}{i\pi^{D/2}} k^\mu k^\nu k^\rho k^\sigma f(k^2) &= \frac{1}{D(D+2)} \left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \int \frac{d^Dk}{i\pi^{D/2}} (k^2)^2 f(k^2).
\end{aligned}$$

The generalisation to arbitrary higher tensor structures is obvious. From the master formula: a factor k^2 in the numerator is equivalent (appart from prefactors) to a shift in the dimension $D \to D+2$. Let us introduce an operator $\mathbf{D}^+$, which shifts the dimension:

$$\mathbf{D}^+ \int \frac{d^D k}{i\pi^{D/2}} f\left(k^2\right) \quad = \quad \int \frac{d^{(D+2)} k}{i\pi^{(D+2)/2}} f\left(k^2\right)$$

Shifting the loop momentum like in $k' = k - xp$ introduces the (Feynman or Schwinger) parameters x_j in the numerator. For the tensor reduction it is convenient to work temporarily with Schwinger parameters. Recall:

$$\frac{1}{(-k^2)^\nu} \quad = \quad \frac{1}{\Gamma(\nu)} \int\limits_0^\infty dx\, x^{\nu-1} \exp(xk^2).$$

A Schwinger parameter x in the numerator is equivalent to raising the power of the original propagator by one unit: $\nu \to \nu+1$. It is convenient to denote by $\mathbf{i}^+$ the operator, which raises the power of propagator i by one.

$$\nu_i \mathbf{i}^+ \frac{1}{\left(-k_i^2\right)^{\nu_i}} = \nu_i \frac{1}{\left(-k_i^2\right)^{\nu_i+1}} = \frac{1}{\Gamma(\nu_i)} \int\limits_0^\infty dx_i\, x_i^{\nu_i-1} x_i \exp(x_i k^2).$$

All Schwinger integrals are rewritten in terms of these scalar integrals. Therefore, using an intermediate Schwinger parametrisation, we have expressed all tensor integrals in terms of scalar integrals. The price we paid is that these scalar integrals involve higher powers of the propagators and/or have shifted dimensions. Each integral can be specified by its topology, its value for the dimension D and a set of indices, denoting the powers of the propagators. In general the number of different integrals is quite large.

6.2 Expansion of transcendental functions

To eliminate powers of the loop momentum in the numerator one can trade the loop momentum in the numerator for scalar integrals (e.g. numerator = 1) with higher powers of the propagators and shifted dimensions:

$$\int \frac{d^{2m-2\varepsilon} k_1}{(2\pi)^{2m-2\varepsilon}} \frac{1}{\left(k_1^2\right)^{\nu_1} \left(k_2^2\right)^{\nu_2} \left(k_3^2\right)^{\nu_3}}.$$

In the second step an integral of type (1) is now converted into an infinite sum. Introducing Feynman parameters, performing the momentum integration and then the integration over the Feynman parameters one obtains

$$\int \frac{d^{2m-2\varepsilon} k_1}{i\pi^{m-\varepsilon}} \frac{1}{(-k_1^2)^{\nu_1}} \frac{1}{(-k_2^2)^{\nu_2}} \frac{1}{(-k_3^2)^{\nu_3}}$$

$$
\begin{aligned}
&= \left(-p_{123}^2\right)^{m-\varepsilon-\nu_{123}} \frac{\Gamma(\nu_{123}-m+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \int_0^1 da\, a^{\nu_2-1}(1-a)^{\nu_3-1} \\
&\quad \times \int_0^1 db\, b^{m-\varepsilon-\nu_{23}-1}(1-b)^{m-\varepsilon-\nu_1-1}\left[1-a(1-x)\right]^{m-\varepsilon-\nu_{123}} \\
&= \left(-p_{123}^2\right)^{m-\varepsilon-\nu_{123}} \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(m-\varepsilon-\nu_1)\Gamma(m-\varepsilon-\nu_{23})}{\Gamma(2m-2\varepsilon-\nu_{123})} \\
&\quad \times \sum_{n=0}^{\infty} \frac{\Gamma(n+\nu_2)\Gamma(n-m+\varepsilon+\nu_{123})}{\Gamma(n+1)\Gamma(n+\nu_{23})}(1-x)^n,
\end{aligned}
$$

where $x = p_{12}^2/p_{123}^2$, $\nu_{23} = \nu_2+\nu_3$ and $\nu_{123} = \nu_1+\nu_2+\nu_3$. To arrive at the last line of (1) one expands $[1-a(1-x)]^{m-\varepsilon-\nu_{123}}$ according to

$$
(1-z)^{-c} = \frac{1}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{\Gamma(n+c)}{\Gamma(n+1)} z^n.
$$

Then all Feynman parameter integrals are of the form

$$
\int_0^1 da\, a^{\mu-1}(1-a)^{\nu-1} = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.
$$

The infinite sum in the last line of (1) is a hypergeometric function, where the small parameter ε occurs in the Gamma-functions:

$$
\begin{aligned}
&{}_2F_1(\nu_2, -m+\varepsilon+\nu_{123}; \nu_{23}; 1-x) = \\
&\quad \frac{\Gamma(\nu_{23})}{\Gamma(\nu_2)\Gamma(-m+\varepsilon+\nu_{123})} \sum_{n=0}^{\infty} \frac{\Gamma(n+\nu_2)\Gamma(n-m+\varepsilon+\nu_{123})}{\Gamma(n+1)\Gamma(n+\nu_{23})}(1-x)^n,
\end{aligned}
$$

6.3 Nested Sums

In this section I review the underlying mathematical structure for the systematic expansion of transcendental functions like the hypergeometric function in (1). I discuss properties of particular forms of nested sums, which are called Z-sums and show that they form a Hopf algebra. This Hopf algebra admits as additional structures a conjugation and a convolution product. Z-sums are defined by

$$
Z(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{n \geq i_1 > i_2 > ... > i_k > 0} \frac{x_1^{i_1}}{{i_1}^{m_1}} \cdots \frac{x_k^{i_k}}{{i_k}^{m_k}}.
$$

k is called the depth of the Z-sum and $w = m_1 + ... + m_k$ is called the weight. If the sums go to Infinity ($n = \infty$) the Z-sums are multiple polylogarithms:

$$
Z(\infty; m_1, ..., m_k; x_1, ..., x_k) = \mathrm{Li}_{m_1,...,m_k}(x_1, ..., x_k).
$$

For $x_1 = ... = x_k = 1$ the definition reduces to the Euler-Zagier sums:

$$Z(n;m_1,...,m_k;1,...,1) \;=\; Z_{m_1,...,m_k}(n).$$

For $n = \infty$ and $x_1 = ... = x_k = 1$ the sum is a multiple ζ-value:

$$Z(\infty;m_1,...,m_k;1,...,1) \;=\; \zeta(m_1,...,m_k).$$

The multiple polylogarithms contain as the notation already suggests as subsets the classical polylogarithms $\mathrm{Li}_n(x)$, as well as Nielsen's generalised polylogarithms

$$S_{n,p}(x) \;=\; \mathrm{Li}_{n+1,1,...,1}(x,\underbrace{1,...,1}_{p-1}),$$

and the harmonic polylogarithms

$$H_{m_1,...,m_k}(x) \;=\; \mathrm{Li}_{m_1,...,m_k}(x,\underbrace{1,...,1}_{k-1}).$$

The usefulness of the Z-sums lies in the fact, that they interpolate between multiple polylogarithms and Euler-Zagier sums.

In addition to Z-sums, it is sometimes useful to introduce as well S-sums. S-sums are defined by

$$S(n;m_1,...,m_k;x_1,...,x_k) \;=\; \sum_{n\geq i_1\geq i_2\geq ...\geq i_k\geq 1} \frac{x_1^{i_1}}{{i_1}^{m_1}} \cdots \frac{x_k^{i_k}}{{i_k}^{m_k}}.$$

The S-sums reduce for $x_1 = ... = x_k = 1$ (and positive m_i) to harmonic sums:

$$S(n;m_1,...,m_k;1,...,1) \;=\; S_{m_1,...,m_k}(n).$$

The S-sums are closely related to the Z-sums, the difference being the upper summation boundary for the nested sums: $(i-1)$ for Z-sums, i for S-sums. The introduction of S-sums is redundant, since S-sums can be expressed in terms of Z-sums and vice versa. It is however convenient to introduce both Z-sums and S-sums, since some properties are more naturally expressed in terms of Z-sums while others are more naturally expressed in terms of S-sums.

The Z-sums form an algebra. The unit element in the algebra is given by the empty sum

$$e \;=\; Z(n).$$

The empty sum $Z(n)$ equals 1 for non-negative integer n. Before I discuss the multiplication rule, let me note that the basic building blocks of Z-sums are expressions of the form

$$\frac{x_j^n}{n^{m_j}},$$

which will be called "letters". For fixed n, one can multiply two letters with the same n:

$$\frac{x_1^n}{n^{m_1}} \cdot \frac{x_2^n}{n^{m_2}} \quad = \quad \frac{(x_1 x_2)^n}{n^{m_1+m_2}},$$

e.g. the x_j's are multiplied and the degrees are added. Let us call the set of all letters the alphabet A. As a short-hand notation I will in the following denote a letter just by $X_j = x_j^n/n^{m_j}$. A word is an ordered sequence of letters, e.g.

$$W \quad = \quad X_1, X_2, ..., X_k.$$

The word of length zero is denoted by e. The Z-sums defined in (1) are therefore completely specified by the upper summation limit n and a word W. A quasi-shuffle algebra $\mathcal{A}$ on the vectorspace of words is defined by

$$\begin{aligned} e \circ W &= W \circ e = W, \\ (X_1, W_1) \circ (X_2, W_2) &= X_1, (W_1 \circ (X_2, W_2)) + X_2, ((X_1, W_1) \circ W_2) \\ &\quad + (X_1 \cdot X_2), (W_1 \circ W_2). \end{aligned}$$

Note that "$\cdot$" denotes multiplication of letters as defined in eq. (1), whereas "$\circ$" denotes the product in the algebra $\mathcal{A}$, recursively defined in eq. (1). This defines a quasi-shuffle product for Z-sums. The recursive definition in (1) translates for Z-sums into

$$\begin{aligned} & Z_{m_1,...,m_k}(n) \times Z_{m'_1,...,m'_l}(n) \\ &= \sum_{i_1=1}^{n} \frac{1}{i_1^{m_1}} Z_{m_2,...,m_k}(i_1-1) Z_{m'_1,...,m'_l}(i_1-1) \\ &\quad + \sum_{i_2=1}^{n} \frac{1}{i_2^{m'_1}} Z_{m_1,...,m_k}(i_2-1) Z_{m'_2,...,m'_l}(i_2-1) \\ &\quad + \sum_{i=1}^{n} \frac{1}{i^{m_1+m'_1}} Z_{m_2,...,m_k}(i-1) Z_{m'_2,...,m'_l}(i-1). \end{aligned}$$

The proof that Z-sums obey the quasi-shuffle algebra is sketched in Fig. 8. The outermost sums of the Z-sums on the l.h.s of (1) are split into the three regions indicated in Fig. 8. A simple example for the multiplication of two Z-sums is

$$\begin{aligned} & Z(n;m_1;x_1) Z(n;m_1;x_2) = \\ & \quad Z(n;m_1,m_2;x_1,x_2) + Z(n;m_2,m_1;x_2,x_1) + Z(n;m_1+m_2;x_1x_2). \end{aligned}$$

The quasi-shuffle algebra $\mathcal{A}$ is isomorphic to the free polynomial algebra on the Lyndon words. If one introduces a lexicographic ordering on the letters of the alphabet A, a Lyndon word is defined by the property

$$W < V$$

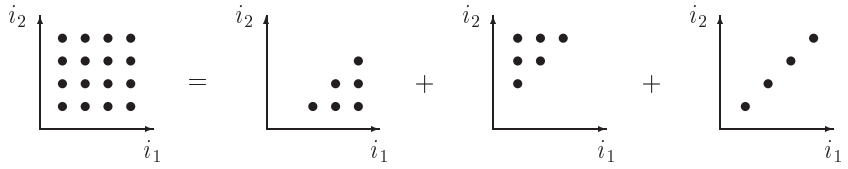
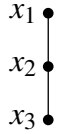


Figure 8: Sketch of the proof for the multiplication of Z-sums. The sum over the square is replaced by the sum over the three regions on the r.h.s.

for any subwords U and V such that $W = U,V$. Here U,V means just concatenation of U and V.

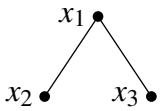
The Z-sums form actuall a Hopf algebra. It is convenient to phrase the coalgebra structure in terms of rooted trees. Z-sums can be represented as rooted trees without any sidebranchings. As a concrete example the pictorial representation of a sum of depth three reads:

$$Z(n;m_1,m_2,m_3;x_1,x_2,x_3) \quad = \quad \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1-1} \sum_{i_3=1}^{i_2-1} \frac{x_1^{i_1}}{{i_1}^{m_1}} \frac{x_2^{i_2}}{{i_2}^{m_2}} \frac{x_3^{i_3}}{{i_3}^{m_3}} \quad = \quad$$



The outermost sum corresponds to the root. By convention, the root is always drawn on the top. Trees with sidebranchings are given by nested sums with more than one subsum, for example:

$$\sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} Z(i-1;m_2,x_2) Z(i-1;m_3;x_3) \quad =$$



Of course, due to the multiplication formula, trees with sidebranchings can always be reduced to trees without any sidebranchings. The coalgebra structure is now formulated in terms of rooted trees. I first introduce some notation how to manipulate rooted trees, following the notation of Kreimer and Connes. An elementary cut of a rooted tree is a cut at a single chosen edge. An admissible cut is any assignment of elementary cuts to a rooted tree such that any path from any vertex of the tree to the root has at most one elementary cut. An admissible cut maps a tree t to a monomial in trees $t_1 \circ \ldots \circ t_{k+1}$. Note that precisely one of these subtrees t_j will contain the root of t. Denote this distinguished tree by $R^C(t)$, and the monomial delivered by the k other factors by $P^C(t)$. The counit $\bar{e}$ is given by

$$\begin{aligned} \bar{e}(e) &= 1, \\ \bar{e}(t) &= 0, \quad t \neq e. \end{aligned}$$

The coproduct Δ is defined by the equations

$$\Delta(e) \quad = \quad e \otimes e,$$

$$\begin{aligned}\Delta(t) &= e\otimes t + t\otimes e + \sum_{\text{adm. cuts } C \text{ of } t} P^C(t)\otimes R^C(t),\\ \Delta(t_1\circ\ldots\circ t_k) &= \Delta(t_1)(\circ\otimes\circ)\ldots(\circ\otimes\circ)\Delta(t_k).\end{aligned}$$

The antipode $\mathcal{S}$ is given by

$$\begin{aligned}\mathcal{S}(e) &= e,\\ \mathcal{S}(t) &= -t - \sum_{\text{adm. cuts } C \text{ of } t} \mathcal{S}\left(P^C(t)\right)\circ R^C(t),\\ \mathcal{S}(t_1\circ\ldots\circ t_k) &= \mathcal{S}(t_1)\circ\ldots\circ\mathcal{S}(t_k).\end{aligned}$$

Since the multiplication in the algebra is commutative the antipode satisfies

$$\mathcal{S}^2 = \text{id}.$$

Let me give some examples for the coproduct and the antipode for Z-sums:

$$\begin{aligned}\Delta Z(n;m_1;x_1) &= e\otimes Z(n;m_1;x_1) + Z(n;m_1;x_1)\otimes e,\\ \Delta Z(n;m_1,m_2;x_1,x_2) &= e\otimes Z(n;m_1,m_2;x_1,x_2) + Z(n;m_1,m_2;x_1,x_2)\otimes e\\ &\quad + Z(n;m_2;x_2)\otimes Z(n;m_1;x_1),\end{aligned}$$

$$\begin{aligned}\mathcal{S}Z(n;m_1;x_1) &= -Z(n;m_1;x_1),\\ \mathcal{S}Z(n;m_1,m_2;x_1,x_2) &= Z(n;m_2,m_1;x_2,x_1) + Z(n;m_1+m_2;x_1x_2).\end{aligned}$$

The Hopf algebra of nested sums has additional structures if we allow expressions of the form

$$\frac{x_0^n}{n^{m_0}} Z(n;m_1,\ldots,m_k;x_1,\ldots,x_k),$$

e.g. Z-sums multiplied by a letter. Then the following convolution product

$$\sum_{i=1}^{n-1} \frac{x^i}{i^m} Z(i-1;\ldots)\,\frac{y^{n-i}}{(n-i)^{m'}} Z(n-i-1;\ldots)$$

can again be expressed in terms of expressions of the form (1). An example is

$$\begin{aligned}&\sum_{i=1}^{n-1} \frac{x^i}{i} Z_1(i-1)\,\frac{y^{n-i}}{(n-i)} Z_1(n-i-1) =\\ &\quad \frac{x^n}{n}\left[Z\left(n-1;1,1,1;\frac{y}{x},\frac{x}{y},\frac{y}{x}\right) + Z\left(n-1;1,1,1;\frac{y}{x},1,\frac{x}{y}\right)\right.\\ &\quad \left. + Z\left(n-1;1,1,1;1,\frac{y}{x},1\right)\right] + (x\leftrightarrow y)\,.\end{aligned}$$

In addition there is a conjugation, e.g. sums of the form

$$-\sum_{i=1}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) (-1)^i\,\frac{x^i}{i^m} S(i;\ldots)$$

can also be reduced to terms of the form (1). Although one can easily convert between the notations for S-sums and Z-sums, expressions involving a conjugation tend to be shorter when expressed in terms of S-sums. The name conjugation stems from the following fact: To any function $f(n)$ of an integer variable n one can define a conjugated function $C \circ f(n)$ as the following sum

$$C \circ f(n) \;=\; \sum_{i=1}^{n} \binom{n}{i} (-1)^i f(i).$$

Then conjugation satisfies the following two properties:

$$\begin{aligned} C \circ 1 &= 1, \\ C \circ C \circ f(n) &= f(n). \end{aligned}$$

An example for a sum involving a conjugation is

$$-\sum_{i=1}^{n} \binom{n}{i} (-1)^i \, \frac{x^i}{i} S_1(i) = S\left(n; 1, 1; 1-x, \frac{1}{1-x}\right) - S\left(n; 1, 1; 1-x, 1\right).$$

Finally there is the combination of conjugation and convolution, e.g. sums of the form

$$-\sum_{i=1}^{n-1} \binom{n}{i} (-1)^i \, \frac{x^i}{i^m} S(i;\ldots) \, \frac{y^{n-i}}{(n-i)^{m'}} S(n-i;\ldots)$$

can also be reduced to terms of the form (1). An example is given by

$$\begin{aligned} &-\sum_{i=1}^{n-1} \binom{n}{i} (-1)^i \, S(i;1;x) \, S(n-i;1;y) = \\ &\quad \frac{1}{n} \left\{ S(n;1;y) + (1-x)^n \left[S\left(n;1;\frac{1}{1-\frac{1}{x}}\right) - S\left(n;1;\frac{1-\frac{y}{x}}{1-\frac{1}{x}}\right) \right] \right\} \\ &\quad + \frac{(-1)^n}{n} \left\{ S(n;1;x) + (1-y)^n \left[S\left(n;1;\frac{1}{1-\frac{1}{y}}\right) - S\left(n;1;\frac{1-\frac{x}{y}}{1-\frac{1}{y}}\right) \right] \right\}. \end{aligned}$$

6.4 Expansion of hypergeometric functions

In this section I discuss how the algebraic tools introduced in the previous section can be used to solve the problems outlined at the end of Sect. 6.2. First I give some motivation for the introduction of Z-sums: The essential point is that Z-sums interpolate between multiple polylogarithms and Euler-Zagier-sums, such that the interpolation is compatible with the algebra structure. On

the one hand, we expect multiple polylogarithm to appear in the Laurent expansion of the transcendental functions (1), a fact which is confirmed a posteriori. Therefore it is important that multiple polylogarithms are contained in the class of Z-sums. On the other the expansion parameter ε occurs in the functions (1) inside the arguments of Gamma-functions. The basic formula for the expansion of Gamma-functions reads

$$\begin{aligned}\Gamma(n+\varepsilon) = \Gamma(1+\varepsilon)\Gamma(n)\Big[1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)\\ +\varepsilon^3 Z_{111}(n-1)+...+\varepsilon^{n-1}Z_{11...1}(n-1)\Big],\end{aligned}$$

containing Euler-Zagier sums for finite n. As a simple example I discuss the expansion of

$$\sum_{i=0}^{\infty}\frac{\Gamma(i+a_1+t_1\varepsilon)\Gamma(i+a_2+t_2\varepsilon)}{\Gamma(i+1)\Gamma(i+a_3+t_3\varepsilon)}x^i$$

into a Laurent series in ε. Here a_1, a_2 and a_3 are assumed to be integers. Up to prefactors the expression in (1) is a hypergeometric function ${}_2F_1$. Using $\Gamma(x+1)=x\Gamma(x)$, partial fractioning and an adjustment of the summation index one can transform (1) into terms of the form

$$\sum_{i=1}^{\infty}\frac{\Gamma(i+t_1\varepsilon)\Gamma(i+t_2\varepsilon)}{\Gamma(i)\Gamma(i+t_3\varepsilon)}\frac{x^i}{i^m},$$

where m is an integer. Now using (1) one obtains

$$\Gamma(1+\varepsilon)\sum_{i=1}^{\infty}\frac{(1+\varepsilon t_1 Z_1(i-1)+...)(1+\varepsilon t_2 Z_1(i-1)+...)}{(1+\varepsilon t_3 Z_1(i-1)+...)}\frac{x^i}{i^m}.$$

Inverting the power series in the denominator and truncating in ε one obtains in each order in ε terms of the form

$$\sum_{i=1}^{\infty}\frac{x^i}{i^m}Z_{m_1...m_k}(i-1)Z_{m'_1...m'_l}(i-1)Z_{m''_1...m''_n}(i-1)$$

Using the quasi-shuffle product for Z-sums the three Euler-Zagier sums can be reduced to single Euler-Zagier sums and one finally arrives at terms of the form

$$\sum_{i=1}^{\infty}\frac{x^i}{i^m}Z_{m_1...m_k}(i-1),$$

which are harmonic polylogarithms $H_{m,m_1,...,m_k}(x)$. This completes the algorithm for the expansion in ε for sums of the form (1). Since the one-loop integral discussed in (1) is a special case of (1), this algorithm also applies to the integral (1). In addition, this algorithm shows that in the expansion of hypergeometric functions ${}_{J+1}F_J(a_1,...,a_{J+1};b_1,...,b_J;x)$ around integer values of the parameters a_k and b_l only harmonic polylogarithms appear in the result.

Let me come back to the example of the one-loop Feynman integral discussed in Sect. 6.2. For $\nu_1 = \nu_2 = \nu_3 = 1$ and $m = 2$ in (1) one obtains:

$$\begin{aligned}
&\int \frac{d^{4-2\varepsilon}k_1}{i\pi^{2-\varepsilon}} \frac{1}{(-k_1^2)} \frac{1}{(-k_2^2)} \frac{1}{(-k_3^2)} \\
&= \frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{\left(-p_{123}^2\right)^{-1-\varepsilon}}{1-x} \sum_{n=1}^{\infty} \varepsilon^{n-1} H_{\underbrace{1,...,1}_{n}}(1-x).
\end{aligned}$$

Here, all harmonic polylogarithms can be expressed in terms of Nielsen polylogarithms, which in turn simplify to powers of the standard logarithm:

$$H_{\underbrace{1,...,1}_{n}}(1-x) = S_{0,n}(1-x) = \frac{(-1)^n}{n!} (\ln x)^n .$$

This particular example is very simple and one recovers the well-known all-order result

$$\frac{\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{\left(-p_{123}^2\right)^{-1-\varepsilon}}{\varepsilon^2} \frac{1-x^{-\varepsilon}}{1-x},$$

which (for this simple example) can also be obtained by direct integration.

6.5 The integral representation of multiple polylogarithms

The multiple polylogarithms are special cases of Z-sums. They are obtained from Z-sums by taking the outermost sum to infinity:

$$Z(\infty; m_1, ..., m_k; x_1, ..., x_k) = \mathrm{Li}_{m_1,...,m_k}(x_1, ..., x_k).$$

Being special cases of Z-sums they obey the quasi-shuffle Hopf algebra for Z-sums. Multiple polylogarithms have been defined in this article via the sum representation (1). In addition, they admit an integral representation. From this integral representation a second algebra structure arises, which turns out to be a shuffle Hopf algebra. To discuss this second Hopf algebra it is convenient to introduce for $z_k \neq 0$ the following functions

$$G(z_1, ..., z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

In this definition one variable is redundant due to the following scaling relation:

$$G(z_1, ..., z_k; y) = G(xz_1, ..., xz_k; xy)$$

If one further defines

$$g(z; y) = \frac{1}{y - z},$$

then one has

$$\frac{d}{dy}G(z_1,...,z_k;y) \;=\; g(z_1;y)G(z_2,...,z_k;y)$$

and

$$G(z_1,z_2,...,z_k;y) \;=\; \int_0^y dt\; g(z_1;t)G(z_2,...,z_k;t).$$

One can sligthly enlarge the set and define $G(0,...,0;y)$ with k zeros for z_1 to z_k to be

$$G(0,...,0;y) \;=\; \frac{1}{k!}(\ln y)^k.$$

This permits us to allow trailing zeros in the sequence $(z_1,...,z_k)$ by defining the function G with trailing zeros via (1) and (1). To relate the multiple polylogarithms to the functions G it is convenient to introduce the following short-hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) \;=\; G(\underbrace{0,...,0}_{m_1-1},z_1,...,z_{k-1},\underbrace{0...,0}_{m_k-1},z_k;y)$$

Here, all z_j for $j=1,...,k$ are assumed to be non-zero. One then finds

$$\mathrm{Li}_{m_1,...,m_k}(x_1,...,x_k) \;=\; (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},...,\frac{1}{x_1...x_k};1\right).$$

The inverse formula reads

$$G_{m_1,...,m_k}(z_1,...,z_k;y) \;=\; (-1)^k\,\mathrm{Li}_{m_1,...,m_k}\left(\frac{y}{z_1},\frac{z_1}{z_2},...,\frac{z_{k-1}}{z_k}\right).$$

Eq. (1) together with (1) and (1) defines an integral representation for the multiple polylogarithms. To make this more explicit I first introduce some notation for iterated integrals

$$\int_0^\Lambda \frac{dt}{t-a_n}\circ...\circ\frac{dt}{t-a_1} \;=\; \int_0^\Lambda \frac{dt_n}{t_n-a_n}\int_0^{t_n}\frac{dt_{n-1}}{t_{n-1}-a_{n-1}}\times...\times\int_0^{t_2}\frac{dt_1}{t_1-a_1}$$

and the short hand notation:

$$\int_0^\Lambda \left(\frac{dt}{t}\circ\right)^m\frac{dt}{t-a} \;=\; \int_0^\Lambda \underbrace{\frac{dt}{t}\circ...\frac{dt}{t}}_{m\,\text{times}}\circ\frac{dt}{t-a}.$$

The integral representation for $\mathrm{Li}_{m_k,...,m_1}(x_k,...,x_1)$ reads then

$$\mathrm{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k \int_0^1 \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{t-b_1}$$
$$\circ \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{t-b_2} \circ ... \circ \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{t-b_k},$$

where the b_j's are related to the x_j's

$$b_j \;=\; \frac{1}{x_1 x_2 ... x_j}.$$

From the iterated integral representation (1) a second algebra structure for the functions $G(z_1,...,z_k;y)$ (and through (1) also for the multiple polylogarithms) is obtained as follows: We take the z_j's as letters and call a sequence of ordered letters $w = z_1,...,z_k$ a word. Then the function $G(z_1,...,z_k;y)$ is uniquely specified by the word $w = z_1,...,z_k$ and the variable y. The neutral element e is given by the empty word, equivalent to

$$G(;y) \;=\; 1.$$

A shuffle algebra on the vector space of words is defined by

$$\begin{aligned} e \circ w &= w \circ e = w, \\ (z_1,w_1) \circ (z_2,w_2) &= z_1,(w_1 \circ (z_2,w_2)) + z_2,((z_1,w_1) \circ w_2). \end{aligned}$$

Note that this definition is very similar to the definition of the quasi-shuffle algebra (1), except that the third term in (1) is missing. In fact, a shuffle algebra is a special case of a quasi-shuffle algebra, where the product of two letters is degenerate: $X_1 \cdot X_2 = 0$ for all letters X_1 and X_2 in the notation of Sect. 6.3. The definition of the shuffle product (1) translates into the following recursive definition of the product of two G-functions:

$$\begin{aligned} &G(z_1,...,z_k;y) \times G(z_{k+1},...,z_n;y) = \\ &\quad \int_0^y \frac{dt}{t-z_1} G(z_2,...,z_k;t) G(z_{k+1},...,z_n;t) \\ &\quad + \int_0^y \frac{dt}{t-z_{k+1}} G(z_1,...,z_k;t) G(z_{k+2},...,z_n;t) \end{aligned}$$

For the discussion of the coalgebra part for the functions $G(z_1,...,z_k;y)$ we may proceed as in Sect. 6.3 and associate to any function $G(z_1,...,z_k;y)$ a rooted tree without sidebranchings as in the following example:

$$G(z_1,z_2,z_3;y) \;=\; \begin{array}{l} z_1 \bullet \\ \quad | \\ z_2 \bullet \\ \quad | \\ z_3 \bullet \end{array}$$

The outermost integration (involving z_1) corresponds to the root. The formulae for the coproduct (1) and the antipode (1) apply then also to the functions $G(z_1,...,z_k;y)$.

A shuffle algebra is simpler than a quasi-shuffle algebra and one finds for a shuffle algebra besides the recursive definitions of the product, the coproduct and the antipode also closed formulae for these operations. For the product one has

$$G(z_1,...,z_k;y)\,G(z_{k+1},...,z_{k+l};y) \;=\; \sum_{shuffle} G\left(z_{\sigma(1)},...,z_{\sigma(k+l)};y\right),$$

where the sum is over all permutations which preserve the relative order of the strings $z_1,...,z_k$ and $z_{k+1},...,z_{k+l}$. This explains the name "shuffle product". For the coproduct one has

$$\Delta G(z_1,...,z_k;y) \;=\; \sum_{j=0}^{k} G(z_1,...,z_j;y)\otimes G(z_{j+1},...,z_k;y)$$

and for the antipode one finds

$$\mathcal{S}G(z_1,...,z_k;y) \;=\; (-1)^k G(z_k,...,z_1;y).$$

The shuffle multiplication is commutative and the antipode satisfies therefore

$$\mathcal{S}^2 \;=\; \mathrm{id}.$$

From (1) this is evident.

6.6 The antipode and integration-by-parts

Integration-by-parts has always been a powerful tool for calculations in particle physics. By using integration-by-parts one may obtain an identity between various G-functions. The starting point is as follows:

$$\begin{aligned}
G(z_1,...,z_k;y) &= \int_0^y dt\left(\frac{\partial}{\partial t}G(z_1;t)\right)G(z_2,...,z_k;t)\\
&= G(z_1;y)G(z_2,...,z_k;y) - \int_0^y dt\; G(z_1;t)g(z_2;t)G(z_3,...,z_k;t)\\
&= G(z_1;y)G(z_2,...,z_k;y) - \int_0^y dt\left(\frac{\partial}{\partial t}G(z_2,z_1;t)\right)G(z_3,...,z_k;t).
\end{aligned}$$

Repeating this procedure one arrives at the following integration-by-parts identity:

$$\begin{aligned}
&G(z_1,...,z_k;y)+(-1)^k G(z_k,...,z_1;y)\\
&\quad= G(z_1;y)G(z_2,...,z_k;y) - G(z_2,z_1;y)G(z_3,...,z_k;y)+...\\
&\qquad -(-1)^{k-1}G(z_{k-1},...z_1;y)G(z_k;y),
\end{aligned}$$

which relates the combination $G(z_1,...,z_k;y)+(-1)^k G(z_k,...,z_1;y)$ to G-functions of lower depth. This relation is useful in simplifying expressions. Eq. (1) can also be derived in a different way. In a Hopf algebra we have for any non-trivial element w the following relation involving the antipode:

$$\sum_{(w)} w^{(1)} \cdot \mathcal{S}(w^{(2)}) \quad = \quad 0.$$

Here Sweedler's notation has been used. Sweedler's notation writes the coproduct of an element w as

$$\Delta(w) \quad = \quad \sum_{(w)} w^{(1)} \otimes w^{(2)}.$$

Working out the relation (1) for the shuffle algebra of the functions $G(z_1,...,\, z_k;y)$, we recover (1).

We may now proceed and check if (1) provides also a non-trivial relation for the quasi-shuffle algebra of Z-sums. This requires first some notation: A composition of a positive integer k is a sequence $I=(i_1,...,i_l)$ of positive integers such that $i_1+...i_l=k$. The set of all composition of k is denoted by $\mathcal{C}(k)$. Compositions act on Z-sums as

$$\begin{aligned}
&(i_1,...,i_l)\circ Z(n;m_1,...,m_k;x_1,...,x_k) \\
&\quad = \quad Z\big(n;m_1+...+m_{i_1},m_{i_1+1}+...+m_{i_1+i_2},...,m_{i_1+...+i_{l-1}+1}+... \\
&\qquad +m_{i_1+...+i_l};x_1...x_{i_1},x_{i_1+1}...x_{i_1+i_2},...,x_{i_1+...+i_{l-1}+1}...x_{i_1+...+i_l}\big)\,,
\end{aligned}$$

e.g. the first i_1 letters of the Z-sum are combined into one new letter, the next i_2 letters are combined into the second new letter, etc.. With this notation for compositions one obtains the following closed formula for the antipode in the quasi-shuffle algebra:

$$\mathcal{S}Z(n;m_1,...,m_k;x_1,...,x_k) \quad = \quad (-1)^k \sum_{I\in\mathcal{C}(k)} I\circ Z(n;m_k,...,m_1;x_k,...,x_1)$$

From (1) we then obtain

$$\begin{aligned}
&Z(n;m_1,...,m_k;x_1,...,x_k)+(-1)^k Z(n;m_k,...,m_1;x_k,...,x_1) \\
&\quad = \quad -\sum_{adm.\ cuts} P^C(Z(n;m_1,...,m_k;x_1,...,x_k)) \\
&\qquad \cdot\mathcal{S}\left(R^C(Z(n;m_1,...,m_k;x_1,...,x_k))\right) \\
&\qquad -(-1)^k \sum_{I\in\mathcal{C}(k)\setminus(1,1,...,1)} I\circ Z(n;m_k,...,m_1;x_k,...,x_1).
\end{aligned}$$

Again, the combination $Z(n;m_1,...,m_k;x_1,...,x_k)+(-1)^k Z(n;m_k,...,m_1;x_k,\ ...,x_1)$ reduces to Z-sums of lower depth, similar to (1). We therefore obtained an "integration-by-parts" identity for objects, which don't have an integral representation. We first observed, that for the G-functions,

which have an integral representation, the integration-by-parts identites are equal to the identities obtained from the antipode. After this abstraction towards an algebraic formulation, one can translate these relations to cases, which only have the appropriate algebra structure, but not necessarily a concrete integral representation. As an example we have

$$\begin{aligned}
&Z(n;m_1,m_2,m_3;x_1,x_2,x_3) - Z(n;m_3,m_2,m_1;x_3,x_2,x_1) = \\
&\quad Z(n;m_1;x_1)Z(n;m_2,m_3;x_2,x_3) - Z(n;m_2,m_1;x_2,x_1)Z(n;m_3;x_3) \\
&\quad -Z(n;m_1+m_2;x_1x_2)Z(n;m_3;x_3) + Z(n;m_2+m_3,m_1;x_2x_3,x_1) \\
&\quad +Z(n;m_3,m_1+m_2;x_3,x_1x_2) + Z(n;m_1+m_2+m_3;x_1x_2x_3),
\end{aligned}$$

which expresses the combination of the two Z-sums of depth 3 as Z-sums of lower depth. The analog example for the shuffle algebra of the G-function reads:

$$G(z_1,z_2,z_3;y) - G(z_3,z_2,z_1;y) \;=\; G(z_1;y)G(z_2,z_3;y) - G(z_2,z_1;y)G(z_3;y).$$

Multiple polylogarithms obey both the quasi-shuffle algebra and the shuffle algebra. Therefore we have for multiple polylogarithms two relations, which are in general independent.

6.7 Numerical evaluation of multiple polylogarithms

The real part of the dilogarithm $\mathrm{Li}_2(x)$ is numerically evaluated as follows: Using the relations

$$\begin{aligned}
\mathrm{Li}_2(x) &= -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x), \\
\mathrm{Li}_2(x) &= -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}\left(\ln(-x)\right)^2,
\end{aligned}$$

the argument is shifted into the range $-1 \le x \le 1/2$. Then

$$\begin{aligned}
\mathrm{Li}_2(x) &= \sum_{i=0}^{\infty} \frac{B_i}{(i+1)!} z^{i+1} \\
&= B_0 z + \frac{B_1}{2} z^2 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n+1)!} z^{2n+1},
\end{aligned}$$

with $z = -\ln(1-x)$ and the B_i are the Bernoulli numbers. The Bernoulli numbers B_i are defined through the generating function

$$\frac{t}{e^t - 1} \;=\; \sum_{i=0}^{\infty} B_n \frac{t^n}{n!}.$$

It is also convenient to use the Clausen function $\mathrm{Cl}_2(x)$ as an auxilary function.

6.8 Mellin-Barnes integrals

$$(A_1+A_2)^{-a} = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} d\sigma \frac{\Gamma(-\sigma)\Gamma(\alpha+\sigma)}{\Gamma(\alpha)} A_1^{\sigma} A_2^{-\alpha-\sigma}$$

with $|\arg A_1 - \arg A_2| < \pi$. The contour is such that the poles of $\Gamma(-\sigma)$ (UV-poles) are to the right and the poles of $\Gamma(\alpha+\sigma)$ (IR-poles) are to the left.

Closing the contour to the left:

$$\frac{1}{2\pi i}\int d\sigma \ldots = \sum_{res} \ldots$$

Closing the contour to the right (negative "umlauf" number):

$$\frac{1}{2\pi i}\int d\sigma \ldots = -\sum_{res} \ldots$$

Barnes first lemma:

$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} dz \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

if none of the poles of $\Gamma(a+z)\Gamma(b+z)$ coincides with the ones from $\Gamma(c-z)\Gamma(d-z)$ and if the contour separates the increasing series of poles from the decreasing ones.

Barnes second lemma:

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} dz \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)} \\ &= \frac{\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)\Gamma(a+e)\Gamma(b+e)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}\end{aligned}$$

7 From differential geometry to Yang-Mills theory

Additional textbooks:

- O. Forster, Analysis 3, Vieweg, (differential forms)

- M. Schottenloher, Geometrie und Symmetrie in der Physik, Vieweg,

- M. Nakahara, Geometry, Topology and Physics, Institute of Physics

7.1 Manifolds

M is an n-dimensional differentiable manifold if

- M is a topological space
- M is provided with a family of open sets $\{U_i\}$ together with corresponding mappings φ_i, such that the family $\{U_i\}$ covers M, that is,

$$\cup U_i \quad = \quad M$$

  and where φ_i is a homeomorphism from U_i onto an open subset $V_i \subseteq \mathbb{R}^n$.
- Given U_i and U_j with $U_i \cap U_j \neq \emptyset$, the map $\varphi_{ij} = \varphi_i \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

The pair (U_i, φ_i) is called a chart, while the whole family $\{(U_i, \varphi_i)\}$ is called an atlas. The subset U_i is called the coordinate neighbourhood while φ_i is the coordinate function or simply the coordinate. Note that M is locally Euclidean and in each coordinate neighbourhood M looks like an open subset of $\mathbb{R}^n$. But note that we do not require that M be $\mathbb{R}^n$ globally.

A map $f : X \to Y$ between two topological spaces X and Y is called a homeomorphism if it is continous and has an inverse $f^{-1} : Y \to X$ which is also continous. In that case X and Y are said to be homeomorphic to each other.

A map $f : X \to Y$ is called a diffeomorphism if it is a homeomorphism and $f^i \in C^\infty$.

Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M \subset \mathbb{R}^n$ a differentiable map. A tangent vector to M at the point $\gamma(t_0)$ is defined by

$$\frac{d}{dt}\gamma(t)\bigg|_{t_0} \in \mathbb{R}^n.$$

The set of all tangent vectors to M at the point p is denoted by T_pM. We denote by T_p^*M the dual vector space of T_pM, i.e. the set of all linear forms

$$\phi : T_pM \to \mathbb{R}.$$

Elements of $\phi \in T_p^* M$ are called co-tangent vectors.

A vector field is a map

$$X : M \to \mathbb{R}^n$$

X associates to each point $p \in M$ a tangent vector $X(p) \in \mathbb{R}^n$.

7.2 Differential forms

A differential form of order one is a map

$$\omega : M \to \bigcup_p T_p^* M$$

with $\omega(p) \in T_p^* M$. The differential form ω associates to each point $p \in M$ a co-tangent vector $\omega(p) \in T_p^* M$. We denote the value of $\omega(p)$ applied to the tangent vector $v \in T_p M$ by

$$\langle \omega(p), v \rangle$$

Example: Let $U \subset \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a differentiable function. The total differential df of f is defined by

$$\langle df(p), v \rangle = \sum_{i=1}^{n} \frac{\partial f(p)}{\partial x_i} v_i.$$

Coordinate representation: Every differential form of order one can be written as

$$\omega = \sum_{i=1}^{n} f_i(x) dx_i.$$

Integrals along a curve: Let $\gamma : [a,b] \to U$ be a curve . The the integral of ω along γ is defined by

$$\int_\gamma \omega = \int_a^b \langle \omega(\gamma(t), \gamma'(t) \rangle dt.$$

Wedge product of linear forms: Let ω_1, ..., $\omega_K \in V^*$ denote linear forms. The wedge product is a map

$$\omega_1 \wedge ... \wedge \omega_k : V^k \to \mathbb{R}$$

defined by

$$(\omega_1 \wedge ... \wedge \omega_k)(v_1, ..., v_k) = \det \begin{pmatrix} \langle \omega_1, v_1 \rangle & ... & \langle \omega_1, v_k \rangle \\ ... & ... & ... \\ \langle \omega_k, v_1 \rangle & ... & \langle \omega_k, v_k \rangle \end{pmatrix}$$

Coordinate representation of differential forms of order k:

$$\omega = \frac{1}{k!}\sum f_{i_1\ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Pull-back of differential forms: Let $U \subset \mathbb{R}^n$ and let

$$\omega = \frac{1}{k!}\sum f_{i_1\ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

be a k-form in U. Further assume that an open set $V \subset \mathbb{R}^m$ and a differential map

$$\varphi = (\varphi_1, \ldots, \varphi_n) : V \to U$$

are given. Then it is possible to define a k-form $\varphi^*\omega$ in V by

$$\varphi^*\omega = \frac{1}{k!}\sum (f_{i_1\ldots i_k} \circ \varphi)\, d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_k}.$$

Remark: k-forms can be integrated over k-dimensional (sub-) manifolds.

Example:

$$A = i\frac{e}{\hbar c} A_\mu(x) dx^\mu,$$

defines a one-form. Further

$$\begin{aligned} dA &= d\left(i\frac{e}{\hbar c} A_\nu dx^\nu\right) = i\frac{e}{\hbar c}\partial_\mu A_\nu dx^\mu \wedge dx^\nu \\ &= i\frac{e}{\hbar c}\frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu \wedge dx^\nu. \end{aligned}$$

This motivates the definition of the field strength two-form

$$F = dA = i\frac{e}{\hbar c}\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

7.3 Riemannian geometry

We consider the transformation from a coordinate system x^0, x^1, x^2, x^3 to the coordinate system x'^0, x'^1, x'^2, x'^3:

$$x^\mu = f^\mu(x'^0, x'^1, x'^2, x'^3).$$

Under this transformation the differentials of the coordinates transform according to

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu.$$

A contra-variant four-vector A^μ is a set of four quantities, which transforms under a coordinate transformation as these differentials:

$$A^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A'^\nu.$$

Let ϕ be a scalar function. The derivatives $\partial\phi/\partial x^\mu$ transform under a change of coordinate systems as

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial \phi}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu}.$$

A co-variant four-vector A_μ is a set of four quantities, which transforms under a coordinate transformation as the derivatives of a scalar function:

$$A_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu$$

A tangent vector can be expressed at every point as a linear combination of basis vectors $\hat{e}_\mu$:

$$V = V^\mu \hat{e}_\mu.$$

For the basis vectors the notation

$$\partial_\mu = \hat{e}_\mu.$$

is often used. A vector field associates to every point of a manifold a vector.

The dual of a vector field is a one-form. A one-form associates at every point of the manifold to a vector a (real or complex) number. A basis for the space of one-forms is given by the differentials dx^μ:

$$\omega = \omega_\mu dx^\mu.$$

The duality between vector fields and one-forms implies

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu.$$

A tensor field with r contra-variant and s co-variant indices maps at the point $x \in M$ r co-tangent vectors and s tangent vectors to a real (or complex) number.

$$\begin{aligned} (T^r_s)_x \quad : \quad & (T^*_x M)^r \times (T_x M)^s \to \mathbb{R}, \\ & \omega^1, ..., \omega^r, V_1, ..., V_s \to (T^r_s)_x \left(\omega^1, ..., \omega^r, V_1, ..., V_s\right). \end{aligned}$$

Coordinate representation:

$$t^{\mu_1,...,\mu_r}_{\nu_1,...,\nu_s}(x) = (T^r_s)_x (dx^{\mu_1}, ..., dx^{\mu_r}, \partial_{\nu_1}, ..., \partial_{\nu_s}).$$

Representation of the tensor field in the basis:

$$T_s^r = \sum_{\mu_1,\ldots,\mu_r=0}^{D-1} \sum_{\nu_1,\ldots,\nu_s=0}^{D-1} t^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) \left(\partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_r}\right) \otimes \left(dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_s}\right).$$

Example: A $(0,2)$-tensor field is defined by

$$g = \sum_{\mu,\nu=0}^{D-1} g_{\mu\nu}(x) dx^\mu \otimes dx^\nu.$$

Definition of a Riemannian manifold: Let M be a differentiable manifold. A Riemannian metric g on M is a type $(0,2)$-tensor field on M which satisfies the following axioms at each point $x \in M$:

$$\begin{aligned} g_x(U,V) &= g_x(V,U) \\ g_x(U,U) &\geq 0 \text{ where the equality holds only when } U = 0 \end{aligned}$$

Here $U, V \in T_xM$ and $g_x = g|_x$.
In short, g_x is a symmetric positive-definite bilinear form.

A metric is called semi-Riemannian metric, if

$$\begin{aligned} g_x(U,V) &= g_x(V,U), \\ \text{and if } g_x(U,V) &= 0 \text{ holds for all } U \in T_xM, \text{ then } V = 0. \end{aligned}$$

A manifold with a semi-Riemannian metric is called a semi-Riemannian manifold.

Remark: Since the metric is symmetric, all eigenvalues of $g_{\mu\nu}$ are real. For a Riemannian manifold all eigenvalues are positive. A semi-Riemannian manifold may have in addition also negative eigenvalues. A manifold with exactly one positive eigenvalue of $g_{\mu\nu}$ (and $(D-1)$ negative eigenvalues) is called Lorentz manifold.

Let (U,φ) be a chart in M and $\{x^\mu\}$ the coordinates. The metric is written as

$$g_p = g_{\mu\nu}(p) dx^\mu \otimes dx^\nu$$

where Einstein's summation convention has been used.

The inverse of $g_{\mu\nu}$ is denoted by $g^{\mu\nu}$.

$$g_{\mu\rho} g^{\rho\nu} = g^{\nu\rho} g_{\rho\mu} = \delta_\mu^\nu$$

The metric gives rise to an isomorphism between T_xM and T_x^*M expressed by

$$\omega_\mu = g_{\mu\nu} U^\nu \qquad U^\mu = g^{\mu\nu} \omega_\nu$$

where $U^\mu \in T_xM$ and $\omega_\mu \in T_x^*M$.

Summary

Differential form:

$$\omega = \omega_\mu dx^\mu$$

Vector field:

$$X = X^\mu \partial_\mu$$

Differential forms of order r:

$$\omega = \omega_{\mu_1\mu_2\ldots\mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$$

In general: tensor fields:

$$T^r_s = \sum_{\mu_1,\ldots,\mu_r=0}^{D-1} \sum_{\nu_1,\ldots,\nu_s=0}^{D-1} t^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) \left(\partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_r}\right) \otimes \left(dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_s}\right).$$

Metric: $(0,2)$-tensor field

$$g = \sum_{\mu,\nu=0}^{D-1} g_{\mu\nu}(x) dx^\mu \otimes dx^\nu.$$

The metric induces an isomorphism between T_xM and T^*_xM expressed by

$$\omega_\mu = g_{\mu\nu} X^\nu$$

Example for a differential form:

$$A = i\frac{e}{\hbar c} A_\mu(x) dx^\mu,$$

This defines a one-form. We have further

$$\begin{aligned} dA &= d\left(i\frac{e}{\hbar c} A_\nu dx^\nu\right) = i\frac{e}{\hbar c} \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\ &= i\frac{e}{\hbar c}\frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu \wedge dx^\nu. \end{aligned}$$

This motivates to define the field strength two-form as

$$F = dA = i\frac{e}{\hbar c}\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

7.4 Hodge theory

Let M be a m-dimensional manifold. If M is endowed with a metric, there is a natural isomorphism between the space of all r-forms and the space of all $(m-r)$ forms, given by the Hodge-operator $*$.

$$\begin{aligned} * \quad : \quad & \Omega^r(M) \to \Omega^{m-r}(M) \\ & *(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}) = \frac{\sqrt{|g|}}{(m-r)!} \varepsilon^{\mu_1 \ldots \mu_r}{}_{\nu_{r+1} \ldots \nu_m} dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_m} \end{aligned}$$

Remark:

$$** \omega \quad = \quad \pm \omega.$$

The sign depends on the signature of the metric. In particular

$$** \omega \quad = \quad \begin{cases} (-1)^{r(m-r)} \omega & \text{Euclidean manifold} \\ (-1)^{r(m-r)+1} \omega & \text{Lorentz manifold} \end{cases}$$

With the help of the Hodge operator one defines a scalar product between two r-forms. Let

$$\begin{aligned} \omega &= \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}, \\ \eta &= \frac{1}{r!} \eta_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}. \end{aligned}$$

We define

$$\begin{aligned} (\omega, \eta) &= \int_M \omega \wedge * \eta \\ &= \frac{1}{r!} \int_M \omega_{\mu_1 \ldots \mu_r} \eta^{\mu_1 \ldots \mu_r} \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^m \end{aligned}$$

This scalar product is symmetric:

$$(\omega, \eta) \quad = \quad (\eta, \omega)$$

Example:

$$*F \quad = \quad * \left(i \frac{e}{\hbar c} \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \right) = \frac{1}{4} i \frac{e}{\hbar c} F^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma = \left(i \frac{e}{\hbar c} \right) \frac{1}{2} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu.$$

We have further

$$(F, F) \quad = \quad \frac{1}{2} \left(i \frac{e}{\hbar c} \right)^2 \int d^4x F_{\mu\nu} F^{\mu\nu}$$

and therefore

$$\int d^4x \, \mathcal{L} \quad = \quad \frac{1}{8\pi} \left(\frac{\hbar c}{e} \right)^2 (F, F).$$

7.5 The covariant derivative

Motivation: In flat space-times the derivatives of a vector

$$\frac{\partial}{\partial x^\nu} A_\mu$$

form a tensor. However, in curved space-times this is no longer true: The derivatives of a vector do not transform as a rank two tensor. Taking the derivative implies comparing a vector field at two different points.

Definition of an affine connection: An affine connection ∇ is a map

$$\begin{aligned} \nabla \quad : \quad & \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M) \\ & (X,Y) \to \nabla_X Y, \end{aligned}$$

which satisfies the following conditions

$$\begin{aligned} \nabla_{(X+Y)} Z &= \nabla_X Z + \nabla_Y Z \\ \nabla_{(fX)} Y &= f \nabla_X Y \\ \nabla_X (Y+Z) &= \nabla_X Y + \nabla_X Z \\ \nabla_X (fY) &= X(f) Y + f \nabla_X Y \end{aligned}$$

where $f \in F(M)$ and $X, Y, Z \in \mathrm{Vect}(M)$.
Take a chart (U, φ) with the coordinate $x = \varphi(p)$ and define D^3 functions called the connection coefficients $C^\mu_{\nu\lambda}$ by

$$\nabla_{e_\mu} e_\nu = e_\lambda C^\lambda_{\mu\nu}$$

where $\{e_\mu\} = \{\partial/\partial_\mu\}$ is the coordinate basis in $T_p M$. For functions $f \in F(M)$ one defines

$$\nabla_X f = X(f) = X^\mu \left(\frac{\partial f}{\partial x^\mu} \right)$$

Then $\nabla_X (fY)$ looks exactly like the Leibnitz rule,

$$\nabla_X (fY) = (\nabla_X f) Y + f (\nabla_X Y)$$

We further set for tensors

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$$

In the following we use the notation

$$\nabla_\mu = \nabla_{e_\mu}$$

We have

$$\begin{aligned}\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu e_\nu) = X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} e_\nu + Y^\nu \nabla_\mu e_\nu \right) \\ &= X^\mu \left(\frac{\partial Y^\lambda}{\partial x^\mu} + Y^\nu C^\lambda_{\mu\nu} \right) e_\lambda .\end{aligned}$$

$\nabla_X Y$ is independent of the derivative of X.

$$\nabla_\mu = \nabla_{\hat{e}_\mu}$$

is called the covariant derivative.

Parallel transport: A vector X is said to be parallel transported along a curve given through V if

$$\nabla_V X = 0$$

holds.

Summary

Recall: We already formulated electro-dynamics in the language of differential geometry. There we associated to the gauge potential $A_\mu(x)$ the one-form

$$A \;=\; i\frac{e}{\hbar c}A_\mu(x)dx^\mu,$$

This one-form defines the covariant derivative

$$D_A \;=\; d+A = d+i\frac{e}{\hbar c}A_\mu dx^\mu$$

and the field strength / curvature by

$$F \;=\; D_A^2 = dA + A\wedge A = dA = i\frac{e}{\hbar c}\frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu.$$

$F_{\mu\nu}$ is the usual field strength tensor:

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu.$$

If we introduce the Hodge operation for differential forms

$$*(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}) \;=\; \frac{\sqrt{|g|}}{(m-r)!}\varepsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m}dx^{\nu_{r+1}}\wedge\ldots\wedge dx^{\nu_m},$$

we can write the action for electromagnetic fields as

$$S \;=\; \frac{1}{8\pi c}\left(\frac{\hbar c}{e}\right)^2\int F\wedge *F.$$

7.6 Fibre bundles

Recall: We already considered manifolds. For every point X of a manifold M we considered the tangent space T_xM. For a D-dimensional manifold we have

$$T_xM \;\sim\; \mathbb{R}^D.$$

It is therefore tempting to consider the product space

$$E \;=\; M\times\mathbb{R}^D,$$

together with a projection

$$\begin{aligned}\pi \;:\; & M\times\mathbb{R}^D\to M,\\ & (x,V)\to x.\end{aligned}$$

Let

$$\frac{\partial}{\partial x^\mu} \quad \text{and} \quad \frac{\partial}{\partial y^\mu}$$

be two bases of the tangent space at point x. Then we have for the transformation matrix

$$\frac{\partial y^\nu}{\partial x^\mu} \quad \in \quad GL(D,\mathbb{R}).$$

The group $GL(D,\mathbb{R})$ is called in this context the structure group of E.

This construction can be generalised in the following points:
- Instead of the tangent space, which is attached to every point of the base space, we can consider an arbitrary manifolds attached to every point of the base space.
- The structure group need not be $GL(D,\mathbb{R})$, but can be any Lie group.
- The requirement of a global product structure is very restrictive. This can be relaxed towards a requirement, that this holds only locally, i.e. in a neighbourhood of every point.

A differentiable fibre bundle (E,π,M,F,G) consists of the following elements :

- a differentiable manifold E called the total space
- a differentiable manifold M called the base space
- a differentiable manifold F called the fibre
- a surjection $\pi : E \to M$ called the projection. The inverse image $\pi^{-1}(p) = F_p$ is called the fibre at p.
- a Lie group G called the structure group, which acts on F from the left.
- a set of open coverings $\{U_i\}$ of M with a diffeomorphism

  $$\phi_i : U_i \times F \to \pi^{-1}(U_i)$$

  such that $\pi\phi_i(p,f) = p$.
  The map ϕ_i is called the local trivialisation, since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$.
- If we write $\phi_i(p,f) = \phi_{i,p}(f)$, the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$ we require that

  $$t_{ij}(p) = \phi_{i,p}^{-1}\phi_{j,p} : F \to F$$

  be an element of G, satiesfying the consistency conditions

  $$t_{ii} = \mathrm{id}, t_{ij} = t_{ji}^{-1}, t_{ij}t_{jk} = t_{ik}.$$

  The $\{t_{ij}\}$ are called the transition functions.

Remark: The product structure $U_i \times F$ is required only locally. Therefore a fibre bundle can be twisted (example: Möbius strip).

Special cases of fibre bundles: A vector bundle is a fibre bundle, whose fibre is a vector space.

A principal bundle has a fibre, which is identical with the structure group G. A principal bundle is also often called a G bundle over M and denoted $P(M,G)$.

A section $\sigma : M \to E$ is a smooth map, which satiesfies $\pi\sigma = \mathrm{id}_M$.

7.7 Connections on fibre bundles

If we consider as fibre the tangent space, we obtain a bundle which is known as tangent bundle. A point in the total space is denoted by (x,V), where x is a point of the base space and V denotes a tangent vector. We already considered the problem of comparing tangent vectors V and W attached to the points x and y, respectively. The problem was solved with the introduction of an affine connection. The affine connection was used to define the parallel transport of tangent vectors.

We now consider the corresponding situation for a principal bunlde. First we note, that in this case the fibre is not a vector space, but a Lie group. Never the less, we face the same problem: We consider a point (x,g_0) in the total space. If we move in the base space from x to y, to which point (y,g_1) does this correspond in the total space ?

In order to answer this question we again define parallel transport through a connection.

We consider the tangent space TP of the total space P. Let $u = (x,g_0)$ be a point in the total space of the principal bundle $P(M,G)$ and let G_x be the fibre at $x = \pi(u)$. The vertical subspace V_uP is a subspace of the tangent space T_uP, which is tangent to G_x at u.

A connection one-form $\omega \in \mathrm{g} \otimes T^*P$, which takes values in the Lie algebra g of G, is a projection of T_uP onto the vertical component $V_uP \cong \mathrm{g}$.
We require further

$$\omega_{ug}(R_{g^*}X) = g^{-1}\omega_u(X)g.$$

The horizontal subspace H_uP is defined to be the kernel of ω. Thus ω defines a unique separation of the tangent space T_uP into the vertical subspace V_uP and the horizontal subspace H_uP such that

$$T_uP \quad = \quad H_uP \oplus V_uP.$$

Horizontal lift : Let $P(M,G)$ be a principal bundle and let $\gamma : [0,1] \to M$ be a curve in M. A curve $\tilde{\gamma} : [0,1] \to P$ is said to be a horizontal lift of γ if $\pi\tilde{\gamma} = \gamma$ and the tangent vector to $\tilde{\gamma}(t)$ always

belongs to $H_{\tilde{\gamma}(t)}P$.

A point u_1 is said to be parallel transported from u_0, if there exists a horizontal lift $\tilde{\gamma}$ between u_1 and u_0.

Remark: Therefore the connection defines the horizontal lift and parallel transport.

Pull-back of differential forms: Let $U \subset \mathbb{R}^n$ and let

$$\omega \;=\; \frac{1}{k!}\sum f_{i_1\ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

be a k-form in U. Further consider an open set $V \subset \mathbb{R}^m$ and a smooth map

$$\varphi = (\varphi_1, \ldots, \varphi_n) : V \to U.$$

Then one defines a k-form $\varphi^*\omega$ in V through

$$\varphi^*\omega \;=\; \frac{1}{k!}\sum (f_{i_1\ldots i_k} \circ \varphi)\, d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_k}.$$

Remark: k-forms can be integrated over k-dimensional (sub-) manifolds.

With the help of a section $\sigma : M \to P$ we can pull-back the connection form ω to M:

$$A \;=\; \sigma^*\omega.$$

We use the notation

$$A \;=\; -\left(\frac{g}{\hbar c}\right) iT^a A^a_\mu dx^\mu = \left(\frac{g}{i\hbar c}\right) T^a A^a_\mu dx^\mu$$

Here, g is the coupling constant. In electro-dynamcis g equals the elementary charge e.

Remark: For two sections σ_1 and σ_2 we always have

$$\sigma_2(x) \;=\; \sigma_1(x)U(x),$$

where $U(x)$ is a x-dependent element of the Lie group G. Then we obtain for the local expressions of the connection one-form

$$A_2 \;=\; U^{-1}A_1U + U^{-1}dU$$

This is nothing else than a gauge transformation.

The connection one-form defines now the covariant derivative

$$D_A \;=\; d + A = d - \left(\frac{g}{\hbar c}\right) iT^a A^a_\mu dx^\mu.$$

Remark: This holds for any Lie group G. For electro-dynamics we have the group $U(1)$. In this case the group has only one generator, which we can take as 1. The formula reduces to

$$D_A = d + A = d - \left(\frac{e}{\hbar c}\right) iA_\mu dx^\mu,$$

This formula we encountered already previously.

With the help of the connection one-form and the covariant derivative we define the curvature two-form of the fibre bundle by

$$F = D_A A = dA + A \wedge A.$$

Remark: This definition is in close analogy with the definition of the Riemannian curvature tensor. Also the Riemannian curvature tensor can be calculated through the covariant derivative of the affine connection.

Note that with

$$A_\mu = \frac{g}{i} T^a A^a_\mu$$

we have

$$\begin{aligned} dA &= d\left(A_\nu dx^\nu\right) = \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu \wedge dx^\nu \\ A \wedge A &= A_\mu dx^\mu \wedge A_\nu dx^\nu = \frac{1}{2}\left(A_\mu A_\nu - A_\nu A_\mu\right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\left[A_\mu, A_\nu\right] dx^\mu \wedge dx^\nu \end{aligned}$$

and therefore

$$\begin{aligned} F &= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right]\right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \end{aligned}$$

in agreement with the previous notation.

$$F_{\mu\nu} = \frac{g}{i} T^a F^a_{\mu\nu}$$

In local coordinates one has

$$F = -\frac{1}{2}\left(\frac{g}{\hbar c}\right) iT^a F^a_{\mu\nu} dx^\mu \wedge dx^\nu,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \left(\frac{g}{\hbar c}\right) f^{abc} A^b_\mu A^c_\nu.$$

In this context we also introduce the dual field strength

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

or equivalently, the dual field strength two-form

$$\begin{aligned} *F = * \left(\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu\right) &= \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}dx^\rho \wedge dx^\sigma \\ &= \frac{1}{2}\tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \end{aligned}$$

Then the action can be written as

$$S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{a\,\mu\nu} = \frac{1}{g^2}\int \mathrm{Tr}\, F \wedge *F$$

Substituting the explicit expressions, we recover the previous result:

$$S = \frac{1}{2g^2}\int \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

With the Hodge inner product the action can be written as

$$S = \frac{1}{g^2}\int \mathrm{Tr}\, F \wedge *F = \frac{1}{g^2}\,\mathrm{Tr}\,(F,F) = \frac{1}{g^2}\,\mathrm{Tr}\,||F||^2$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \left(\frac{g}{\hbar c}\right) f^{abc} A^b_\mu A^c_\nu.$$

The Lagrange density is invariant under the following gauge transformation:

$$T^a A^a_\mu(x) \;\rightarrow\; U(x)\left(T^a A^a_\mu(x) + i\frac{\hbar c}{g}\partial_\mu\right)U(x)^\dagger$$

where

$$U(x) = \exp\left(-iT^a\theta^a(x)\right).$$

The Bianchi identity reads

$$\partial_\mu \tilde{F}^{\mu\nu} + \left[A_\mu, \tilde{F}^{\mu\nu}\right] = 0$$

The (classical) Yang-Mills field equations read

$$\partial_\mu F^{\mu\nu} + \left[A_\mu, F^{\mu\nu}\right] = 0$$

Summary:

	Maxwell	Yang-Mills
Bianchi	$\partial_\mu \tilde{F}^{\mu\nu} = 0$	$\partial_\mu \tilde{F}^{\mu\nu} + \left[A_\mu, \tilde{F}^{\mu\nu}\right] = 0$
Euler-Lagrange	$\partial_\mu F^{\mu\nu} = 0$	$\partial_\mu F^{\mu\nu} + \left[A_\mu, F^{\mu\nu}\right] = 0$

7.8 Instantons

Recall from the last lecture:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\, F\wedge *F = \frac{1}{2g^2}\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

$$\begin{aligned} F &= \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu, \quad F_{\mu\nu} = \frac{g}{i}T^a F^a_{\mu\nu}, \\ A &= A_\mu dx^\mu, \quad A_\mu = \frac{g}{i}T^a A^a_\mu. \end{aligned}$$

$$F = dA + A\wedge A.$$

Dual field strength:

$$*F = \frac{1}{2}\tilde{F}_{\mu\nu}dx^\mu\wedge dx^\nu, \quad \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

Bianchi identity:

$$\partial_\mu \tilde{F}^{\mu\nu} + \left[A_\mu, \tilde{F}^{\mu\nu}\right] = 0$$

Euler-Lagrange equations:

$$\partial_\mu F^{\mu\nu} + \left[A_\mu, F^{\mu\nu}\right] = 0$$

Gauge transformations:

$$A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U.$$

In this lecture: $SU(2)$ Yang-Mills theory in euclidean space. The generators of $SU(2)$ are proportional to the Pauli matrices:

$$I^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I^2 = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad I^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let us look at classical solutions for the Yang-Mills theory, such that the action is finite

$$S = \frac{1}{2g^2}\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

In Euclidean space this implies

$$\lim_{|x|\to\infty} F_{\mu\nu}(x) = 0.$$

One is tempted to conclude that this implies

$$\lim_{|x|\to\infty} A_\mu(x) \;=\; 0.$$

But this condition is too strong. In fact, the vanishing of the field strength implies only

$$\lim_{|x|\to\infty} A_\mu(x) \;=\; U^{-1}(x)\partial_\mu U(x),$$

which is obtained from $A_\mu(x) = 0$ by a gauge transformation. A field configuration like $U^{-1}(x)\partial_\mu U(x)$ is called "pure gauge". Therefore we look at classical solutions, which approach a pure gauge configuration at infinity.

For four-dimensional Euclidean space the points at infinity form a three-sphere. The gauge-transformation U at infinity represents therefore a mapping from S^3 to $SU(2)$. Since $SU(2)$ is topologically equivalent to a three-sphere, we look at mappings

$$S^3 \;\to\; S^3.$$

These mappings are characterized by a topological winding number.

Simple example:

$$\begin{aligned} U(1) &\to U(1), \\ e^{i\phi} &\to e^{ni\phi}. \end{aligned}$$

The winding number is given by

$$n \;=\; \frac{1}{16\pi^2}\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

In Euclidean space we have the positivity condition

$$\int d^4x\, \mathrm{Tr}\, \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)\left(F^{\mu\nu} \pm \tilde{F}^{\mu\nu}\right) \;\geq\; 0.$$

Since further

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \;=\; F_{\mu\nu}F^{\mu\nu}$$

we have

$$\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)\left(F^{\mu\nu} \pm \tilde{F}^{\mu\nu}\right) \;=\; 2F_{\mu\nu}F^{\mu\nu} \pm 2F_{\mu\nu}\tilde{F}^{\mu\nu}$$

and therefore

$$\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \;\geq\; \left|\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}\right| = 16\pi^2\,|n|\,.$$

Thus the action satisifes

$$S \geq \frac{8\pi^2|n|}{g^2}.$$

The action is minimized for

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}.$$

These solutions are called – depending on the sign – self-dual and anti-self-dual solutions.

Remark: Self-dual and anti-self-dual configurations are automatically solutions of the Euler-Lagrange euqation. From the Bianchi identity

$$\partial_\mu\tilde{F}^{\mu\nu} + \left[A_\mu, \tilde{F}^{\mu\nu}\right] = 0$$

and $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ it follows immediately

$$\partial_\mu F^{\mu\nu} + \left[A_\mu, F^{\mu\nu}\right] = 0.$$

The instanton solution of Belavin, Polyakov, Schwartz and Tyuplin (Phys. Lett. 59 B, 1975, 85):

$$A_\mu = \frac{x^2}{x^2+\rho^2}U^{-1}(x)\partial_\mu U(x)$$

where ρ is the instanton radius and

$$U(x) = \frac{1}{\sqrt{x^2}}(x_0 + i\vec{x}\vec{\sigma})$$

With the identity

$$(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma}) = (\vec{a}\vec{b})1 + i(\vec{a}\times\vec{b})\vec{\sigma}$$

we obtain

$$(x_0 + i\vec{x}\vec{\sigma})(x_0 - i\vec{x}\vec{\sigma}) = x_0^2 + \vec{x}^2$$

Therefore one shows for

$$U^\dagger(x) = \frac{1}{\sqrt{x^2}}(x_0 - i\vec{x}\vec{\sigma})$$

that

$$U^{-1}(x) = U^\dagger(x).$$

Further

$$\det(x_0 + i\vec{x}\vec{\sigma}) = x^2,$$

and therefore

$$U(x) \in SU(2).$$

In the limit $x^2 \gg \rho^2$ the gauge field reduces to a pure gauge configuration

$$A_\mu \approx U^{-1}(x)\partial_\mu U(x)$$

7.9 Chern classes and Chern characters

Invariant polynomials: Let N be a complex $k \times k$ matrix and $P(N)$ a polynomial in the components of N. $P(N)$ is called an invariant polynomial if

$$P(N) \quad = \quad P\left(g^{-1}Ng\right)$$

for all $g \in GL(k,\mathbb{C})$.

Examples for invariant polynomials:

$$\det(1+N),$$
$$\text{Tr } \exp N.$$

Remark: If A has eigenvalues $\{\lambda_1,...,\lambda_k\}$, $P(A)$ is a symmetric function of the eigenvalues.

For a principal bundle we consider invariant polynomials as a function of

$$\frac{i}{2\pi}F$$

The total Chern class is defined by

$$c(F) \quad = \quad \det\left(1+\frac{iF}{2\pi}\right)$$

Since F is a two-form, $c(F)$ is a direct sum of forms of even degrees,

$$c(F) \quad = \quad 1+c_1(F)+c_2(F)+...$$

where $c_j(F) \in \Omega^{2j}(M)$ is called the jth Chern class.
In an m-dimensional manifold M, the Chern class $c_j(F)$ with $2j > m$ vanishes trivially.
Irrespective of $\dim M$, the series terminates at $c_k(F) = \det(iF/2\pi)$ and $c_j(F) = 0$ for $j > k$.

Example: $SU(2)$ over a four-dimensional manifold with

$$F \quad = \quad F^a I^a, \quad F^a = \frac{1}{2}\frac{g}{i}F^a_{\mu\nu}dx^\mu \wedge dx^\nu.$$

$$\begin{aligned} c(F) &= \det\left(1+\frac{iF}{2\pi}\right) = \det\left(1+\frac{i}{2\pi}F^aI^a\right) = \det\begin{pmatrix} 1+\frac{i}{4\pi}F^3 & \frac{i}{4\pi}\left(F^1-iF^2\right) \\ \frac{i}{4\pi}\left(F^1+iF^2\right) & 1-\frac{i}{4\pi}F^3 \end{pmatrix} \\ &= 1+\frac{1}{16\pi^2}\left(F^1\wedge F^1+F^2\wedge F^2+F^3\wedge F^3\right) \\ &= 1+\frac{1}{8\pi^2}\text{Tr } F\wedge F. \end{aligned}$$

Working out the first few Chern classes we find

$$\begin{aligned} c_0(F) &= 1, \\ c_1(F) &= \frac{i}{2\pi}\mathrm{Tr}\, F, \\ c_2(F) &= \frac{1}{8\pi^2}\left[\mathrm{Tr}\, F\wedge F - \mathrm{Tr}\, F\wedge \mathrm{Tr}\, F\right]. \end{aligned}$$

Remark: For a $SU(n)$ bundle we have $\mathrm{Tr}\, F = 0$ and therefore the Chern classes simplify to

$$\begin{aligned} c_0(F) &= 1, \\ c_1(F) &= 0, \\ c_2(F) &= \frac{1}{8\pi^2}\mathrm{Tr}\, F\wedge F. \end{aligned}$$

The total Chern character is defined by

$$ch(F) = \mathrm{Tr}\, \exp\left(\frac{iF}{2\pi}\right)$$

The jth Chern character $ch_j(F)$ is

$$ch_j(F) = \frac{1}{j!}\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^j$$

If $2j > \dim M$, $ch_j(F)$ vanishes, hence $ch(F)$ is a polynomial of finite order. The first few Chern characters in terms of Chern classes are

$$\begin{aligned} \mathrm{ch}_0(F) &= k, \\ \mathrm{ch}_1(F) &= c_1(F), \\ \mathrm{ch}_2(F) &= \frac{1}{2}\left(c_1(F)^2 - 2c_2(F)\right). \end{aligned}$$

For a $SU(n)$ bundle we obtain

$$\begin{aligned} \mathrm{ch}_0(F) &= k, \\ \mathrm{ch}_1(F) &= 0, \\ \mathrm{ch}_2(F) &= -\frac{1}{8\pi^2}\mathrm{Tr}\, F\wedge F. \end{aligned}$$

An important property of the Chern classes and Chern characters is the fact that they topologically invariant integrals. An example of such an integral we encountered for the winding number:

$$n = \frac{1}{16\pi^2}\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{8\pi^2}\int \mathrm{Tr}\, F\wedge F = \int c_2(F).$$

8 Supersymmetry

8.1 Groups and symmetries of space-time

8.1.1 The Poincaré group

The symmetry properties of four dimensional space-time are described by the Poincaré group. The group elements act on four vectors according to the following transformation law :

$$x^{\mu'} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}.$$

Λ describes rotations in four dimensional space-time (e.g. ordinary rotations on the spacial components plus boosts) whereas a describes translations.
The group multiplication law is given by

$$\{a_1, \Lambda_1\}\{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2\}.$$

The generators of the Poincaré group can be realised as differential operators :

$$\begin{aligned} P_\mu &= i\partial_\mu, \\ M_{\mu\nu} &= i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right). \end{aligned}$$

The algebra of the Poincaré group is given by

$$\begin{aligned} \left[M_{\mu\nu}, M_{\rho\sigma}\right] &= -i\left(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}\right), \\ \left[M_{\mu\nu}, P_\sigma\right] &= i\left(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu\right), \\ \left[P_\mu, P_\nu\right] &= 0. \end{aligned}$$

The Poincaré algebra is a Lie algebra, but it is not semi-simple, since it has an Abelian non-trivial ideal (P_μ).
Casimir operators are M^2 and W^2 where

$$M^2 = P_\mu P^\mu, \qquad W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}.$$

W^μ is called the Lubanski-Pauli vector.

8.1.2 The homogeneous Lorentz group

With the notation

$$\begin{aligned} (B_i) &= (M^{01}, M^{02}, M^{03}), \\ (R_i) &= (M^{23}, M^{31}, M^{12}), \end{aligned}$$

the algebra of the homogeneous Lorentz group is given by

$$\begin{aligned} \left[B_i, B_j\right] &= -i\varepsilon_{ijk}R_k, \\ \left[R_i, R_j\right] &= i\varepsilon_{ijk}R_k, \\ \left[B_i, R_j\right] &= i\varepsilon_{ijk}B_k. \end{aligned}$$

If one defines

$$\begin{aligned} J_i &= \frac{1}{2}(R_i + iB_i), \\ K_i &= \frac{1}{2}(R_i - iB_i), \end{aligned}$$

the algebra can be writen as direct product of two $SU(2)$ algebren :

$$\begin{aligned} \left[J_i, J_j\right] &= -i\varepsilon_{ijk}J_k, \\ \left[K_i, K_j\right] &= i\varepsilon_{ijk}K_k, \\ \left[J_i, K_j\right] &= 0 \end{aligned}$$

8.2 Mixing internal symmetries with space-time symmetries

No-go theorem by Coleman and Mandula[1]: Any Lie group containing the Poincaré group and an internal symmetry group as maximal subgroups is the trivial product of both. In other words, internal symmetry transformations always commute with the Poincaré transformations.
Extended by Haag, Lopuszański and Sohnius[2] to superalgebras, e.g. some generators of the symmetry obey anticommutation rules instead of commutation rules.

8.3 Grassmann algebra

Ordinary number commute:

$$\left[x_i, x_j\right] = 0.$$

The Grassmann algebra consists of anti-commuting numbers

$$\left\{\theta_i, \theta_j\right\} = 0.$$

A Grassmann algebra of n anti-commuting variables $\{\theta_1, ..., \theta_n\}$ can be regarded as a vectorspace over $\mathbb{C}$ or $\mathbb{R}$ with basis

$$\theta_i,\ \theta_i\theta_j,\ \theta_i\theta_j\theta_k,\ ...,\ \theta_1...\theta_n,$$

(with $i < j < k$, etc.) and dimension

$$\sum_i \left(\begin{array}{c} n \\ i \end{array} \right) = 2^n.$$

The differentiation is defined by

$$\frac{\partial}{\partial\theta_j}\left(\theta_1...\theta_j...\theta_m\right) = (-1)^{j-1}\theta_1...\hat{\theta}_j...\theta_m,$$

[1] S. Coleman and J. Mandula, Phys. Rev. 159 (1967), 1251
[2] R. Haag, J. Lopuszański and M. Sohnius, Phys. Lett. B88 (1975), 257

where the hat indicates that the corresponding variable has to be omitted. Note that

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}F \;=\; -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}F.$$

The Taylor expansion of a function $F(\theta)$ depending on a Grassmann variable θ is given by

$$F(\theta) \;=\; F_0 + F_1\theta.$$

The differential $d\theta$ is also a Grassmann variable:

$$\{\theta, d\theta\} \;=\; 0.$$

Integration over a Grassmann variable is defined by

$$\int d\theta = 0, \qquad \int d\theta\theta = 1.$$

Multiple integrals are defined by iteration:

$$\int d\theta_1 d\theta_2 F(\theta_1,\theta_2) \;=\; \int d\theta_1 \left(\int d\theta_2 F(\theta_1,\theta_2)\right).$$

8.4 Sign conventions

The convention for the metric tensor is

$$g_{\mu\nu} \;=\; \mathrm{diag}(+1,-1,-1,-1)$$

2-dimensional antisymmetric tensor :

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{BA} = -\varepsilon_{AB}$$

Furthermore:

$$\varepsilon^{AB} = \varepsilon^{\dot{A}\dot{B}} = \varepsilon_{AB} = \varepsilon_{\dot{A}\dot{B}}.$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4-dimensional σ^μ-matrices

$$\sigma^\mu_{A\dot{B}} = (1,-\vec{\sigma}) \qquad \bar{\sigma}^{\mu\dot{A}B} = (1,\vec{\sigma})$$

Weyl representation for the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Raising and lowering of indices:

$$p^A = \varepsilon^{AB} p_B, \qquad q^{\dot{A}} = \varepsilon^{\dot{A}\dot{B}} q_{\dot{B}},$$
$$p_{\dot{B}} = p^{\dot{A}} \varepsilon_{\dot{A}\dot{B}}, \qquad q_B = q^A \varepsilon_{AB}.$$

Note that raising an index is done by left-multiplication, whereas lowering is performed by right-multiplication.

Spinor products:

$$\begin{aligned} \langle pq \rangle &= p^A q_A \\ [pq] &= p_{\dot{A}} q^{\dot{A}} \end{aligned}$$

The spinor product is anti-symmetric: If p_A and q_B are Weyl spinors (two-component spinors with complex valued entries), then

$$\langle pq \rangle = -\langle qp \rangle, \qquad [pq] = -[qp].$$

Spinor products of Grassmann valued spinors are often denoted as

$$\begin{aligned} \psi\chi &= \psi^A \chi_A, \\ \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{A}} \bar{\chi}^{\dot{A}}. \end{aligned}$$

Note that we have

$$\psi\chi = \chi\psi, \qquad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}.$$

Here we have one sign from the anti-symmetry of the spinor product and another sign from exchanging two Grassmann variables.
Useful relations:

$$\begin{aligned} \bar{\sigma}^{\mu \dot{A} B} &= \varepsilon^{\dot{A}\dot{D}} \varepsilon^{BC} \sigma^{\mu}_{C\dot{D}}, \\ \sigma^{\mu}_{A\dot{B}} &= \bar{\sigma}^{\mu \dot{C} D} \varepsilon_{\dot{C}\dot{B}} \varepsilon_{DA}. \end{aligned}$$

8.5 Superspace

The superspace coordinates are $z = (x_\mu, \theta_A, \bar{\theta}_{\dot{A}})$ where the θ_A and $\bar{\theta}_{\dot{A}}$ $(A, \dot{A} \in \{1,2\})$ are Grassmannian coordinates:

$$\begin{aligned} \{\theta_A, \theta_B\} &= 0, \\ \theta^2 &= \theta^A \theta_A = \theta_B \varepsilon^{AB} \theta_A, \\ \theta_A \psi_B &= -\psi_B \theta_A. \end{aligned}$$

The supersymmetric transformations are generated by

$$\begin{aligned} Q_A &= \frac{\partial}{\partial\theta^A} + i\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\partial_\mu, \\ \bar{Q}^{\dot{A}} &= \frac{\partial}{\partial\bar{\theta}_{\dot{A}}} + i\theta^A\sigma^\mu_{A\dot{B}}\varepsilon^{\dot{B}\dot{A}}\partial_\mu = \frac{\partial}{\partial\bar{\theta}_{\dot{A}}} + i\bar{\sigma}^{\mu\,\dot{A}B}\theta_B\partial_\mu. \end{aligned}$$

They satisfy the supersymmetry algebra:

$$\begin{aligned} &\left\{Q_A, \bar{Q}_{\dot{A}}\right\} = -2i\sigma^\mu_{A\dot{A}}\partial_\mu = -2\sigma^\mu_{A\dot{A}}P_\mu, \\ &\{Q_A, Q_B\} = \left\{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\right\} = 0, \\ &\left[P_\mu, Q_A\right] = \left[P_\mu, \bar{Q}_{\dot{A}}\right] = 0. \end{aligned}$$

The covariant derivatives are given by

$$\begin{aligned} D_\mu &= \partial_\mu, \\ D_A &= \frac{\partial}{\partial\theta^A} - i\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\partial_\mu, \\ \bar{D}^{\dot{A}} &= \frac{\partial}{\partial\bar{\theta}_{\dot{A}}} - i\theta^A\sigma^\mu_{A\dot{B}}\varepsilon^{\dot{B}\dot{A}}\partial_\mu = \frac{\partial}{\partial\bar{\theta}_{\dot{A}}} - i\bar{\sigma}^{\mu\,\dot{A}B}\theta_B\partial_\mu. \end{aligned}$$

By construction they anticommute with the supersymmetry generators

$$\{Q_A, D_B\} = \left\{\bar{Q}_{\dot{A}}, \bar{D}_{\dot{B}}\right\} = \{Q_A, \bar{D}_{\dot{B}}\} = \left\{\bar{Q}_{\dot{A}}, D_B\right\} = 0.$$

Further

$$\begin{aligned} &\{D_A, \bar{D}_{\dot{A}}\} = 2i\sigma^\mu_{A\dot{A}}\partial_\mu, \\ &\{D_A, D_B\} = \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0. \end{aligned}$$

8.6 Supersymmetric fields

A superfield $F(x,\theta,\bar{\theta})$ is expanded according to

$$\begin{aligned} F(x,\theta,\bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\phi}(x) \\ &\quad + \theta^2 m(x) + \bar{\theta}^2 n(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &\quad + \theta^2\bar{\theta}\bar{\psi}(x) + \bar{\theta}^2\theta\psi(x) + \bar{\theta}^2\theta^2 d(x) \end{aligned}$$

It contains four scalars (f, m, n and d), four spinors (ϕ, $\bar{\phi}$, ψ and $\bar{\psi}$) and one complex vector v_μ.

8.6.1 Chiral super-fields

Chiral superfields are defined by

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0, \qquad D_\alpha\bar{\Phi}(x,\theta,\bar{\theta}) = 0.$$

Let

$$y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}, \qquad \bar{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} = x^\mu - i\bar{\theta}\bar{\sigma}^\mu\theta.$$

Then

$$\bar{D}^{\dot{A}} y^\mu = 0, \qquad D_A \bar{y}^\mu = 0.$$

Therefore a chiral superfield $\bar{D}_{\dot{\alpha}}\Phi = 0$ depends only on $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$ and θ:

$$\Phi = \Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y).$$

Use Taylor expansion:

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!}(y-x)^n \partial_x^n f(x), \qquad (y^\mu - x^\mu) = -i\theta\sigma^\mu\bar{\theta}.$$

Therefore

$$\begin{aligned}\Phi &= \phi(x) - i\left(\theta\sigma^\mu\bar{\theta}\right)\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) - i\sqrt{2}\left(\theta\sigma^\mu\bar{\theta}\right)\theta^A\partial_\mu\psi_A(x) + \theta^2 F(x).\end{aligned}$$

Then

$$\begin{aligned}\bar{\Phi} &= \bar{\phi}(x) + i\left(\theta\sigma^\mu\bar{\theta}\right)\partial_\mu\bar{\phi}(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu\bar{\phi}(x) \\ &\quad + \sqrt{2}\bar{\psi}(x)\bar{\theta} + i\sqrt{2}\left(\theta\sigma^\mu\bar{\theta}\right)\partial_\mu\bar{\psi}_{\dot{A}}(x)\bar{\theta}^{\dot{A}} + \bar{\theta}^2\bar{F}(x).\end{aligned}$$

8.6.2 Vector super-fields

A vector superfield is defined by

$$V = V^*$$

The general form of a vector superfield (after expansion in the Grassmann variables) is given by

$$\begin{aligned}V &= C(x) + i\theta\eta(x) - i\bar{\theta}\bar{\eta}(x) + \theta\sigma^\mu\bar{\theta}V_\mu(x) \\ &\quad + \frac{i}{2}\theta\theta\left(M(x) + iN(x)\right) - \frac{i}{2}\bar{\theta}\bar{\theta}\left(M(x) - iN(x)\right) \\ &\quad + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\eta(x)\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\eta}(x)\right) \\ &\quad + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) + \frac{1}{2}\Box C(x)\right)\end{aligned}$$

In the Wess-Zumino gauge we have

$$C = \eta = \bar{\eta} = M = N = 0$$

The vector superfield reduces then to

$$V = \left(\theta\sigma^\mu\bar{\theta}\right)V_\mu + i\theta^2\left(\bar{\theta}\bar{\lambda}\right) - i\bar{\theta}^2\left(\theta\lambda\right) + \frac{1}{2}\theta^2\bar{\theta}^2 D.$$

8.7 Transformation of the fields

Infinitessimal susy-transformations are given by

$$\delta_\eta \;=\; \eta^A Q_A + \bar\eta_{\dot B}\bar Q^{\dot B}$$

A finite susy-transformation is given by

$$\exp i\left(\eta Q + \bar\eta \bar Q\right)$$

Useful relations

$$\begin{aligned} \bar D^{\dot A} y^\mu = 0, \qquad & D_A \bar y^\mu = 0, \\ Q_A y^\mu = 0, \qquad & \bar Q^{\dot A} \bar y^\mu = 0. \end{aligned}$$

as well as

$$\begin{aligned} Q_A &= D_A + 2i\sigma^\mu_{A\dot B}\bar\theta^{\dot B}\partial_\mu, \\ \bar Q^{\dot B} &= \bar D^{\dot B} + 2i\bar\sigma^{\mu\,\dot B C}\theta_C\partial_\mu. \end{aligned}$$

For a chiral superfield

$$\Phi(y,\theta) \;=\; \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y).$$

we have

$$\begin{aligned} \delta_\eta\Phi &= (\eta Q)\,\Phi + 2i\,(\bar\eta\bar\sigma^\mu\theta)\,\partial_\mu\Phi \\ &= \sqrt{2}\eta\psi(y) + 2\eta\theta F(y) + 2i\,(\bar\eta\bar\sigma^\mu\theta)\,\partial_\mu\Phi \\ &= \sqrt{2}\eta\psi(y) + 2\theta\eta F(y) - 2i\theta\sigma^\mu\bar\eta\partial_\mu\phi(y) - \sqrt{2}i\theta^2\bar\eta\bar\sigma^\mu\partial_\mu\psi(y). \end{aligned}$$

Therefore

$$\begin{aligned} \phi' &= \sqrt{2}\eta\psi, \\ \psi' &= -\sqrt{2}i\sigma^\mu\bar\eta\partial_\mu\phi + \sqrt{2}\eta F, \\ F' &= -\sqrt{2}i\bar\eta\bar\sigma^\mu\partial_\mu\psi. \end{aligned}$$

The vector superfield transforms as

$$\begin{aligned} V_\mu^{a\prime} &= i\left(\bar\eta\bar\sigma^\mu\lambda^a - \bar\lambda^a\bar\sigma^\mu\eta\right) = -i\left(\bar\lambda^a\bar\sigma^\mu\eta + \lambda^a\sigma^\mu\bar\eta\right) = i\left(\eta\sigma^\mu\bar\lambda^a + \bar\eta\bar\sigma^\mu\lambda^a\right) \\ \lambda^{a\prime} &= \frac{1}{2}\sigma^\mu\bar\sigma^\nu\eta F^a_{\mu\nu} + i\eta D^a, \\ D^{a\prime} &= \eta\sigma^\mu\partial_\mu\bar\lambda^a + \partial_\mu\lambda^a\sigma^\mu\bar\eta. \end{aligned}$$

8.8 Lagrange density for supersymmetric QCD

To construct supersymmetric Lagrange densities one uses the highets component fields of a superfield, e.g. the F-terms for chiral superfields and the D-terms for vector superfields. Note that the projection onto these components can be written as a differentiation or integration with respect to the Grassmann coordinates:

$$\int d^4\theta \;=\; \frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial\bar{\theta}^2}$$

The Lagrange densitiy for supersymmetric QCD:

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{8g^2}\,\mathrm{Tr}\; WW|_F + \frac{1}{8g^2}\,\mathrm{Tr}\; \bar{W}\bar{W}|_{\bar{F}} \\ & + \bar{\Phi}_+ e^{2gV}\Phi_+\big|_D + \bar{\Phi}_- e^{-2gV^T}\Phi_-\Big|_D + m\Phi_-\Phi_+|_F + m\bar{\Phi}_+\bar{\Phi}_-\big|_{\bar{F}}\,.\end{aligned}$$

The function

$$W(\Phi_+,\Phi_-) \;=\; m\Phi_-\Phi_+$$

is often called the superpotential. In the Lagrange density we have

$$\begin{aligned} W_A &= -\frac{1}{4}\left(\bar{D}\bar{D}\right)\left(e^{-2gV}D_A e^{2gV}\right), \\ \bar{W}_{\dot{A}} &= -\frac{1}{4}\left(DD\right)\left(\left(\bar{D}_{\dot{A}}e^{2gV}\right)e^{-2gV}\right), \\ V &= T^a V^a, \end{aligned}$$

and V^a is given in the Wess-Zumino gauge by

$$V^a \;=\; \left(\theta\sigma^\mu\bar{\theta}\right)V^a_\mu + i\theta^2\left(\bar{\theta}\bar{\lambda}^a\right) - i\bar{\theta}^2\left(\theta\lambda^a\right) + \frac{1}{2}\theta^2\bar{\theta}^2 D^a.$$

We further have

$$\begin{aligned} V^aV^b &= \left(\theta\sigma^\mu\bar{\theta}\right)\left(\theta\sigma^\nu\bar{\theta}\right)V^a_\mu V^b_\nu, \\ V^aV^bV^c &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} e^{2gV} \;=\;& 1 + 2g\left(\theta\sigma^\mu\bar{\theta}\right)V^a_\mu T^a + 2gi\theta^2\left(\bar{\theta}\bar{\lambda}^a\right)T^a - 2gi\bar{\theta}^2\left(\theta\lambda^a\right)T^a \\ & + \theta^2\bar{\theta}^2\left(gD^aT^a + g^2V^a_\mu V^{\mu\,b}T^aT^b\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\Phi}_+ e^{2gV}\Phi_+\big|_D \;=\;& \left(D_\mu\phi_+\right)^\dagger\left(D^\mu\phi_+\right) + i\bar{\psi}_+\bar{\sigma}^\mu D_\mu\psi_+ + i\sqrt{2}g\left[\bar{\phi}_+\lambda^aT^a\psi_+ - \bar{\psi}_+\bar{\lambda}^aT^a\phi_+\right] \\ & + g\bar{\phi}_+D^aT^a\phi_+ + \bar{F}_+F_+,\end{aligned}$$

where the covariant derivative is given by

$$D_\mu = \partial_\mu + igT^a V_\mu^a.$$

For the vector field we find

$$\begin{aligned}
e^{-2gV} D_A e^{2gV} &= 2g\sigma^\mu\bar{\theta}V_\mu + 4gi\theta\left(\bar{\theta}\bar{\lambda}\right) - 2gi\bar{\theta}^2\lambda - ig\bar{\theta}^2\sigma^\mu\bar{\sigma}^\nu\theta\left(\partial_\mu V_\mu - 2igV_\nu V_\mu\right) \\
&\quad + 2\theta\bar{\theta}^2\left(gD + g^2 V_\mu V^\mu\right) - g\theta^2\bar{\theta}^2\sigma^\mu\partial_\mu\bar{\lambda} + 4g^2 i\theta^2\sigma^\mu\bar{\theta}\left[V_\mu, \bar{\theta}\bar{\lambda}\right], \\
\left(\bar{D}_{\dot{A}} e^{2gV}\right) e^{-2gV} &= \varepsilon_{\dot{A}\dot{B}}\left\{2g\bar{\sigma}^\mu\theta V_\mu + 4gi\bar{\theta}\left(\theta\lambda\right) - 2gi\theta^2\bar{\lambda} - ig\theta^2\bar{\sigma}^\mu\sigma^\nu\bar{\theta}\left(\partial_\mu V_\nu + 2igV_\mu V_\nu\right)\right. \\
&\quad \left. - 2\theta^2\bar{\theta}\left(gD + g^2 V_\mu V^\mu\right) - g\theta^2\bar{\theta}^2\bar{\sigma}^\mu\partial_\mu\lambda + 4g^2 i\sigma^\mu\theta\bar{\theta}^2\left[V_\mu, \theta\lambda\right]\right\}.
\end{aligned}$$

$$\begin{aligned}
D^A D_A &= -\varepsilon^{AB}\frac{\partial}{\partial\theta^A}\frac{\partial}{\partial\theta^B} + 2i\varepsilon^{AB}\sigma^\mu_{A\dot{C}}\bar{\theta}^{\dot{C}}\frac{\partial}{\partial\theta^B}\partial_\mu + \bar{\theta}^2\Box, \\
\bar{D}_{\dot{A}}\bar{D}^{\dot{A}} &= \varepsilon_{\dot{A}\dot{B}}\frac{\partial}{\partial\bar{\theta}_{\dot{A}}}\frac{\partial}{\partial\bar{\theta}_{\dot{B}}} - 2i\varepsilon_{\dot{A}\dot{B}}\bar{\sigma}^{\mu\dot{A}C}\theta_C\frac{\partial}{\partial\bar{\theta}_{\dot{B}}}\partial_\mu + \theta^2\Box.
\end{aligned}$$

We obtain for the Lagrange density

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + i\bar{\lambda}^a\bar{\sigma}^\mu\left(D_\mu\lambda\right)^a + \frac{1}{2}D^a D^a \\
&\quad + \left(D_\mu\phi_+\right)^\dagger\left(D^\mu\phi_+\right) + i\bar{\psi}_+\bar{\sigma}^\mu D_\mu\psi_+ + i\sqrt{2}g\left[\bar{\phi}_+\lambda^a T^a\psi_+ - \bar{\psi}_+\bar{\lambda}^a T^a\phi_+\right] \\
&\quad + g\bar{\phi}_+ D^a T^a\phi_+ + \bar{F}_+ F_+ \\
&\quad + \left(\tilde{D}_\mu\phi_-\right)^\dagger\left(\tilde{D}^\mu\phi_-\right) + i\bar{\psi}_-\bar{\sigma}^\mu\tilde{D}_\mu\psi_- - i\sqrt{2}g\left[\bar{\phi}_-\lambda^a\, T^{a\,T}\psi_- - \bar{\psi}_-\bar{\lambda}^a\, T^{a\,T}\phi_-\right] \\
&\quad - g\bar{\phi}_- D^a\, T^{a\,T}\phi_- + \bar{F}_- F_- \\
&\quad + m\left[-\psi_-\psi_+ - \bar{\psi}_+\bar{\psi}_- + \bar{\phi}_+\bar{F}_- + \bar{\phi}_-\bar{F}_+ + \phi_+ F_- + \phi_- F_+\right],
\end{aligned}$$

where

$$\tilde{D}_\mu = \partial_\mu - ig\, T^{a\,T} V^a_\mu$$

Elimination of the D-terms and F-terms:

$$\frac{\partial\mathcal{L}}{\partial D} = \frac{\partial\mathcal{L}}{\partial F} = 0$$

$$\begin{aligned}
\mathcal{L}_D &= \frac{1}{2}D^a D^a + g\bar{\phi}_+ D^a T^a\phi_+ - g\bar{\phi}_- D^a\, T^{a\,T}\phi_- = -\frac{1}{2}g^2\left(\bar{\phi}_+ T^a\phi_+ - \bar{\phi}_-\, T^{a\,T}\phi_-\right)^2 \\
\mathcal{L}_{F_+} &= \bar{F}_+ F_+ + m\phi_- F_+ + m\bar{\phi}_-\bar{F}_+ = -m^2\bar{\phi}_-\phi_- \\
\mathcal{L}_{F_-} &= \bar{F}_- F_- + m\phi_+ F_- + m\bar{\phi}_+\bar{F}_- = -m^2\bar{\phi}_+\phi_+
\end{aligned}$$

The relations are

$$D^a = -g\left(\bar{\phi}_+ T^a\phi_+ - \bar{\phi}_-\, T^{a\,T}\phi_-\right),$$

$$F_+ = -m\bar{\phi}_- \qquad \bar{F}_+ = -m\phi_-$$
$$F_- = -m\bar{\phi}_+ \qquad \bar{F}_- = -m\phi_+$$

Finally:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + i\bar{\lambda}^a\bar{\sigma}^\mu\left(D_\mu\lambda\right)^a \\
&+ \left(D_\mu\phi_+\right)^\dagger\left(D^\mu\phi_+\right) + i\bar{\psi}_+\bar{\sigma}^\mu D_\mu\psi_+ + i\sqrt{2}g\left[\bar{\phi}_+\lambda^a T^a\psi_+ - \bar{\psi}_+\bar{\lambda}^a T^a\phi_+\right] \\
&+ \left(\tilde{D}_\mu\phi_-\right)^\dagger\left(\tilde{D}^\mu\phi_-\right) + i\bar{\psi}_-\bar{\sigma}^\mu\tilde{D}_\mu\psi_- - i\sqrt{2}g\left[\bar{\phi}_-\lambda^a\, T^{a\,T}\,\psi_- - \bar{\psi}_-\bar{\lambda}^a\, T^{a\,T}\,\phi_-\right] \\
&- m\psi_-\psi_+ - m\bar{\psi}_+\bar{\psi}_- - m^2\bar{\phi}_+\phi_+ - m^2\bar{\phi}_-\phi_- - \frac{1}{2}g^2\left(\bar{\phi}_+T^a\phi_+ - \bar{\phi}_-\, T^{a\,T}\,\phi_-\right)^2
\end{aligned}$$

Note that (after one partial integration)

$$i\bar{\psi}_-\bar{\sigma}^\mu\partial_\mu\psi_- \;=\; i\psi_-\sigma^\mu\partial_\mu\bar{\psi}_-$$

Eliminating $T^{a\,T}$ we obtain

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + i\bar{\lambda}^a\bar{\sigma}^\mu\left(D_\mu\lambda\right)^a \\
&+ \left(D_\mu\phi_+\right)^\dagger\left(D^\mu\phi_+\right) + i\bar{\psi}_+\bar{\sigma}^\mu D_\mu\psi_+ + i\sqrt{2}g\left[\bar{\phi}_+\lambda^a T^a\psi_+ - \bar{\psi}_+\bar{\lambda}^a T^a\phi_+\right] \\
&+ \left(D_\mu\bar{\phi}_-\right)^\dagger\left(D^\mu\bar{\phi}_-\right) + i\psi_-\sigma^\mu D_\mu\bar{\psi}_- + i\sqrt{2}g\left[\phi_-\bar{\lambda}^a T^a\bar{\psi}_- - \psi_-\lambda^a T^a\bar{\phi}_-\right] \\
&- m\psi_-\psi_+ - m\bar{\psi}_+\bar{\psi}_- - m^2\bar{\phi}_+\phi_+ - m^2\bar{\phi}_-\phi_- - \frac{1}{2}g^2\left(\bar{\phi}_+T^a\phi_+ - \phi_-T^a\bar{\phi}_-\right)^2
\end{aligned}$$

With

$$\bar{\Psi} = (\psi_-, \bar{\psi}_+), \qquad \Psi = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix},$$
$$\bar{\Lambda} = \left(i\lambda^A, -i\bar{\lambda}_{\dot{A}}\right), \qquad \Lambda = \begin{pmatrix} i\lambda_A \\ -i\bar{\lambda}^{\dot{A}} \end{pmatrix},$$

we obtain

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{i}{2}\bar{\Lambda}^a\gamma^\mu\left(D_\mu\Lambda\right)^a + i\bar{\Psi}\gamma^\mu D_\mu\Psi - m\bar{\Psi}\Psi \\
&+ \left(D_\mu\phi_+\right)^\dagger\left(D^\mu\phi_+\right) - m^2\bar{\phi}_+\phi_+ + \left(D_\mu\bar{\phi}_-\right)^\dagger\left(D^\mu\bar{\phi}_-\right) - m^2\bar{\phi}_-\phi_- \\
&+ \sqrt{2}g\left[\bar{\phi}_+\bar{\Lambda}^a T^a P_+\Psi + \bar{\Psi}P_-\Lambda^a T^a\phi_+ - \phi_-\bar{\Lambda}^a T^a P_-\Psi - \bar{\Psi}P_+\Lambda^a T^a\bar{\phi}_-\right] \\
&- \frac{1}{2}g^2\left(\bar{\phi}_+T^a\phi_+ - \phi_-T^a\bar{\phi}_-\right)^2.
\end{aligned}$$

8.9 Supersymmetry breaking

Q_α and $\bar{Q}_{\dot{\alpha}}$ can be considered as creation ($\bar{Q}_{\dot{\alpha}}$) and annihilation (Q_α) operators. The vacuum is defined by

$$Q_\alpha\,|0\rangle \;=\; 0.$$

Take the trace over

$$\left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu.$$

Then

$$\begin{aligned} H = P_0 &= \frac{1}{4}\left(Q_1\bar{Q}_{\dot{1}} + \bar{Q}_{\dot{1}}Q_1 + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{2}}Q_2\right) \\ &= \frac{1}{4}\left[\left(Q_1 + \bar{Q}_{\dot{1}}\right)\left(Q_1 + \bar{Q}_{\dot{1}}\right) + \left(Q_2 + \bar{Q}_{\dot{2}}\right)\left(Q_2 + \bar{Q}_{\dot{2}}\right)\right] \end{aligned}$$

where $H = P_0$ is the Hamiltonian. The operator on the right-hand side is positive semi-definite. (The operator $Q_\alpha + \bar{Q}_{\dot{\alpha}}$ is a hermitian operator, due to $\bar{Q}_{\dot{\alpha}} = {Q_\alpha}^\dagger$. A hermitian operator has real eigenvalues and the square of these eigenvalues is a non-negative number.) Therefore the linear combination $Q_\alpha + \bar{Q}_{\dot{\alpha}}$ annihilates the vacuum

$$\left(Q_\alpha + \bar{Q}_{\dot{\alpha}}\right)|0\rangle = 0,$$

if and only if the Hamiltonian does as well,

$$\langle 0|H|0\rangle = 0.$$

Therefore the vacuum energy serves as an order parameter: If $\langle 0|H|0\rangle \neq 0$ then the supersymmetry is broken. On the other hand, if supersymmetry is unbroken, then the supercharge annihilates the vacuum. Since the vacuum is defined by $Q_\alpha|0\rangle = 0$, we conclude that if supersymmetry is unbroken, all supercharges annihilate the vacuum :

$$\begin{aligned} Q_\alpha|0\rangle &= 0, \\ \bar{Q}_{\dot{\alpha}}|0\rangle &= 0. \end{aligned}$$

8.10 Supersymmetric relations

After removing the colour factors, QCD at tree-level may be viewed as an effective supersymmetric theory, where the quarks and the gluons form a super-multiplet (a $N = 1$ vector supermultiple). In an unbroken supersymmetric theory, the supercharge

$$Q = \begin{pmatrix} Q_\alpha \\ Q^{\dot{\beta}} \end{pmatrix}$$

annihilates the vacuum, and therefore

$$\langle 0|[Q, \Phi_1\Phi_2...\Phi_n]|0\rangle = \sum_{i=1}^{n} \langle 0|\Phi_1...[Q, \Phi_i]...\Phi_n|0\rangle = 0$$

where the field Φ_i denotes either a gauge boson g or a fermion Λ. It is convenient to multiply the supercharge Q by a Grassmann spinor $\bar{\eta}$. $\bar{\eta}$ is usually chosen to be a Grassmann number θ times a spinor $\bar{u}(q)$ for an arbitrary null-vector q. It is convenient to define

$$Q(q) = \theta\bar{u}(q)Q.$$

The commutators are then given by

$$\begin{aligned}\left[Q(q),g^{\pm}(k)\right] &= \mp\Gamma^{\pm}(k,q)\Lambda^{\pm}(k),\\ \left[Q(q),\Lambda^{\pm}(k)\right] &= \mp\Gamma^{\mp}(k,q)g^{\pm}(k),\end{aligned}$$

with

$$\Gamma^{+}(k,q)=\theta\langle q+|k-\rangle=\theta[qk],\qquad \Gamma^{-}(k,q)=\theta\langle q-|k+\rangle=\theta\langle qk\rangle.$$

The simplest case is the all-equal helicity one. Using the supersymmetric relation

$$\begin{aligned}0 &= \left\langle 0\left|\left[Q,\Lambda_1^{+}g_2^{+}...g_n^{+}\right]\right|0\right\rangle\\ &= -\Gamma^{-}(p_1,q)A(g_1^{+},g_2^{+},...,g_n^{+})+\Gamma^{+}(p_2,q)A(\Lambda_1^{+},\Lambda_2^{+},...,g_n^{+})\\ &\quad +...+\Gamma^{+}(p_n,q)A(\Lambda_1^{+},g_2^{+},...,\Lambda_n^{+}).\end{aligned}$$

Note that $\Gamma(p,q)$ anticommutes with the fermion operators Λ due to the Grassmann nature of θ. Since massless gluinos, like quarks, have only helicity-conserving interactions, all of the amplitudes but the first one must vanish. Therefore so must the like-helicity amplitude

$$A(g_1^{+},g_2^{+},...,g_n^{+}) = 0.$$

Similar, with one negative helicity one gets

$$\begin{aligned}0 &= \left\langle 0\left|\left[Q,\Lambda_1^{+}g_2^{-}g_3^{+}...g_n^{+}\right]\right|0\right\rangle\\ &= -\Gamma^{-}(p_1,q)A(g_1^{+},g_2^{-},g_3^{+},...,g_n^{+})-\Gamma^{-}(p_2,q)A(\Lambda_1^{+},\Lambda_2^{-},g_3^{+}...,g_n^{+}).\end{aligned}$$

Choosing $q=p_2$ one shows that

$$A(g_1^{+},g_2^{-},g_3^{+},...,g_n^{+}) = 0.$$

Setting $q=p_1$ one obtains

$$A(\Lambda_1^{+},\Lambda_2^{-},g_3^{+}...,g_n^{+}) = 0.$$

With two negative helicities one starts to relate non-zero amplitudes:

$$\begin{aligned}0 &= \left\langle 0\left|\left[Q,\Lambda_1^{+}g_2^{+}...g_j^{-}...g_{n-1}^{+}g_n^{-}\right]\right|0\right\rangle\\ &= -\Gamma^{-}(p_1,q)A(g_1^{+},g_2^{+},...,g_j^{-},...,g_{n-1}^{+},g_n^{-})-\Gamma^{-}(p_j,q)A(\Lambda_1^{+},g_2^{+},...,\Lambda_j^{-},...,g_{n-1}^{+},g_n^{-})\\ &\quad -\Gamma^{-}(p_n,q)A(\Lambda_1^{+},g_2^{+},...,g_j^{-},...,g_{n-1}^{+},\Lambda_n^{-}).\end{aligned}$$

setting the reference momentum equal to $q=p_j$ and using the expression for the maximally helicity violating gluon amplitudes one obtains the expression for an amplitude with a pair of quarks:

$$A_n^{tree}(q_1^{+},g_2^{+},...,g_j^{-},...,g_{n-1}^{+},\bar{q}_n^{-}) = i\left(\sqrt{2}\right)^{n-2}\frac{\langle 1j\rangle\langle jn\rangle^3}{\langle 12\rangle\langle 23\rangle...\langle n1\rangle}.$$

8.11 Spontaneous breaking of supersymmetry

8.11.1 The mechanism of O'Raifeartaigh

The simplest model consists of three chiral superfields. In this model the superpotential is given by

$$W(\Phi_1,\Phi_2,\Phi_3) \;=\; \lambda\Phi_1 + m\Phi_2\Phi_3 + g\Phi_1\Phi_2\Phi_2$$

8.11.2 The mechanism of Fayet and Iliopoulos

Fayet and Iliopoulos considered a model of two chiral superfields Φ_1 and Φ_2 and a vector superfield V corresponding to an abelian gauge group. The potential in this model is given by

$$2\kappa\left(F_1F_1^* + F_2F_2^* + \frac{1}{2}D^2\right).$$

8.12 The minimal supersymmetric standard model

The supersymmetric part of the minimal supersymmetric standard model (MSSM) reads:

$$\begin{aligned}
\mathcal{L}_{SUSY} \;=\;& \frac{1}{8g^2}\,\mathrm{Tr}\;W_{SU(2)}W_{SU(2)}\big|_F + \frac{1}{8g'^2}\,\mathrm{Tr}\;W_{U(1)}W_{U(1)}\big|_F + \frac{1}{8g_s^2}\,\mathrm{Tr}\;W_{SU(3)}W_{SU(3)}\big|_F + h.c.\\
&+ \bar{Q}e^{-2g'V'^T-2gV^T-2g_sV_s^T}Q\Big|_D + \bar{U}e^{2g'V'+2gV+2g_sV_s}U\Big|_D + \bar{D}e^{2g'V'+2gV+2g_sV_s}D\Big|_D\\
&+ \bar{L}e^{-2g'V'^T-2gV^T}L\Big|_D + \bar{N}e^{2g'V'+2gV}N\Big|_D + \bar{E}e^{2g'V'+2gV}E\Big|_D\\
&+ \bar{H}_1e^{2g'V'+2gV}H_1\Big|_D + \bar{H}_2e^{2g'V'+2gV}H_2\Big|_D\\
&+ [(\lambda_dH_1QD + \lambda_uH_2QU + \lambda_eH_1LE + \lambda_\nu H_2LN - \mu H_1H_2)|_F + h.c.]
\end{aligned}$$

Remarks: The Higgs boson is in a chiral superfield and has a spin $1/2$ partner, the higgsino. Anomaly cancellation in the fermion triangle loop requires that there are two Higgs superfields.

The supersymmetric part alone would predict equal masses for all particles in a super-multiplet. This is not observed. Therefore one adds additional terms to the Lagrange density, which explicitly break supersymmetry. As these are rather arbitrary, one usually requires that the SUSY-breaking terms

- generate no quadratic divergences,

- are gauge-invariant,

- conserve R-parity

These three conditions define the "soft-breaking-terms".

$$\begin{aligned}\mathcal{L}_{soft} &= -m_{\tilde{q}}^2 |\tilde{q}_L|^2 - m_{\tilde{u}}^2 |\tilde{u}_R|^2 - m_{\tilde{d}}^2 |\tilde{d}_R|^2 - m_{\tilde{l}}^2 |\tilde{l}_L|^2 - m_{\tilde{\nu}}^2 |\tilde{\nu}_R|^2 - m_{\tilde{e}}^2 |\tilde{e}_R|^2 \\ &\quad -m_1^2 |H_1|^2 - m_2^2 |H_2|^2 + \frac{1}{2}\left(M_1 \lambda' \lambda' + M_2 \lambda^a \lambda^a + M_3 \lambda_s^a \lambda_s^a\right) \\ &\quad -\varepsilon_{ij}\left(\lambda_d A_d H_1^i \tilde{q}_L^j \tilde{d}_R^\dagger + \lambda_u A_u H_2^i \tilde{q}_L^j \tilde{u}_R^\dagger + \lambda_e A_e H_1^i \tilde{l}_L^j \tilde{e}_R^\dagger + \lambda_\nu A_\nu H_2^i \tilde{l}_L^j \tilde{\nu}_R^\dagger - m_3^2 H_1^i H_2^j + h.c.\right)\end{aligned}$$

R-parity: Consider a transformation, which transforms the Grassmann coordinates as

$$\theta_A \to e^{-i\varphi} \qquad \bar{\theta}_{\dot{A}} \to e^{i\varphi}\bar{\theta}_{\dot{A}}$$

Furthermore, require that each superfield has a well-defined transformation law under this transformation, like

$$\Phi \to e^{iq_\Phi \varphi}\Phi.$$

q_Φ is called the *R*-charge of the superfield. Writing out the superfield in components

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y),$$

it is clear that the component fields transform with *R*-charge

$$\begin{aligned}\phi &: q_\Phi, \\ \psi &: q_\Phi + 1, \\ F &: q_\Phi + 2.\end{aligned}$$

Assign the *R*-charge $+1$ to all quark and lepton superfields, and *R*-charge 0 to the vector superfields and the two Higgs superfields.

R-parity: Restrict $e^{i\varphi}$ to ± 1.

It then follows with the assignment of the *R*-charges above, that all standard model-like particles have *R*-parity $+1$, whereas all superpartners have *R*-parity -1.

Conservation of *R*-parity implies that at each vertex the product of the *R*-parities of the involved particles evaluates to $+1$. This implies that the supersymmetric particles do always occur in pairs.

Corollar: The lightest supersymmetric particle is stable.