Automated calculations for multi-leg processes

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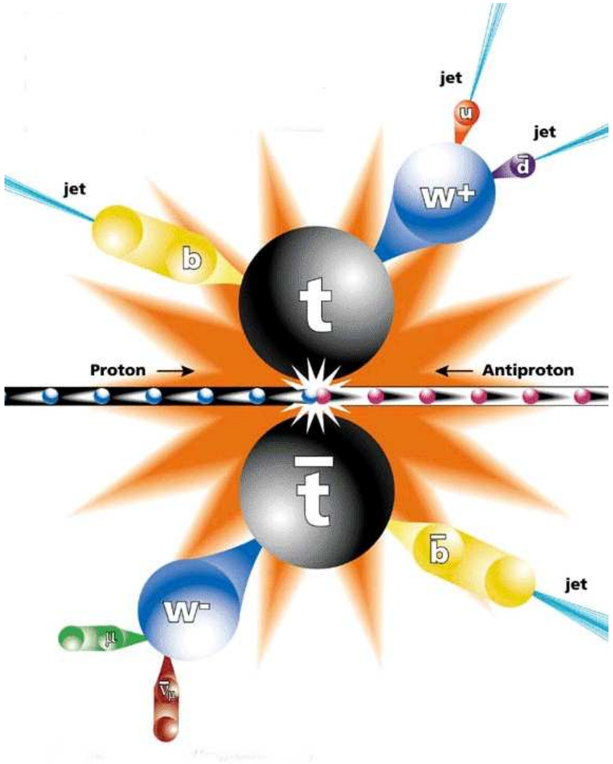
Introduction: **LHC physics**

I.: **Managing lengthy expressions**

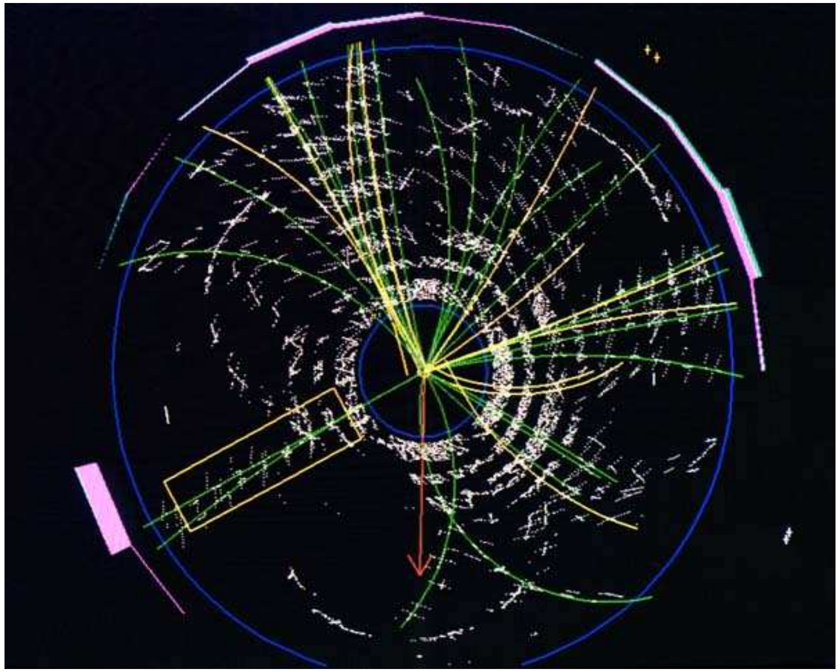
II: **Calculating loop amplitudes**

III: **Cancellation of divergences**

LHC physics



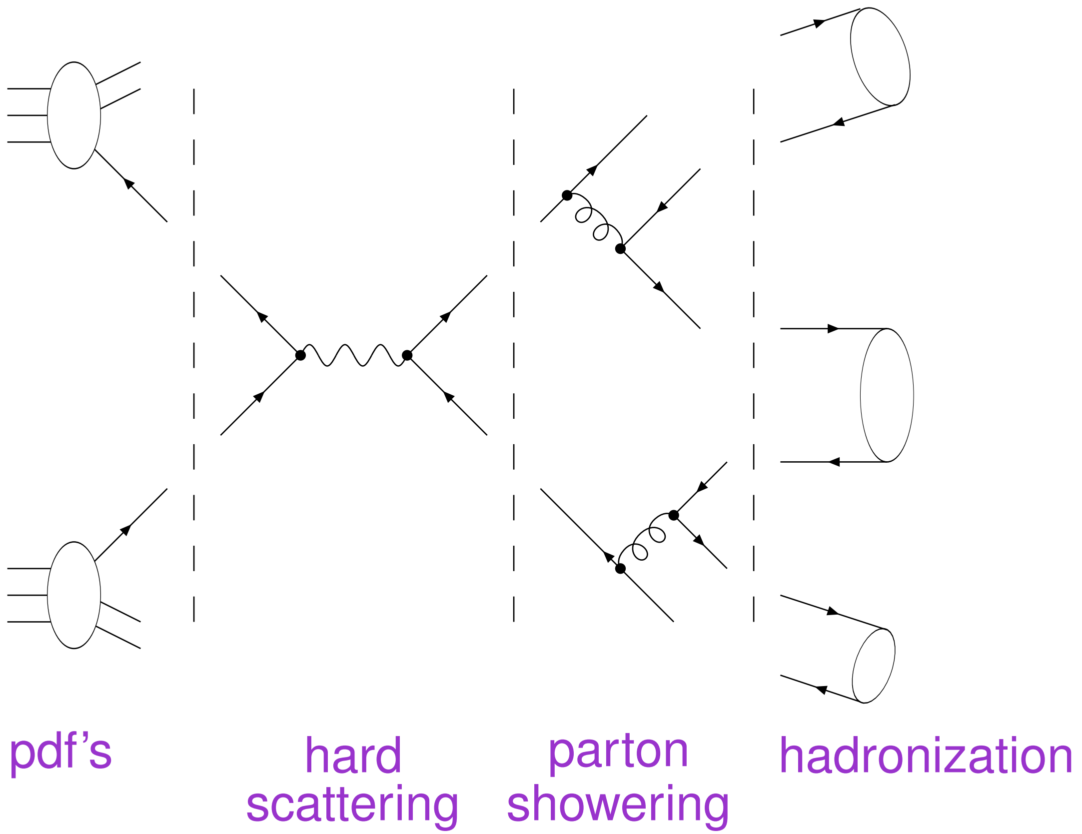
A schematic view of a top-pair event.



A $t\bar{t}$ event from CDF.

Jets: A bunch of particles moving in the same direction

Theoretical understanding



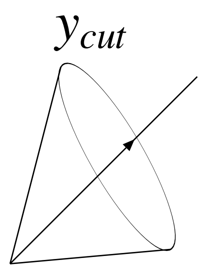
Parton distribution functions are extracted from experiments.
Hard scattering calculated in perturbation theory.
Showering and hadronization depends on approximations and/or models.

Infrared-safe observables depend only mildly on showering and hadronization.

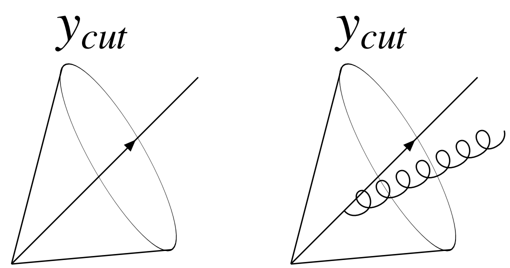
Modeling of jets:

In a perturbative calculation jets are modeled by only a few partons. This improves with the order to which the calculation is done.

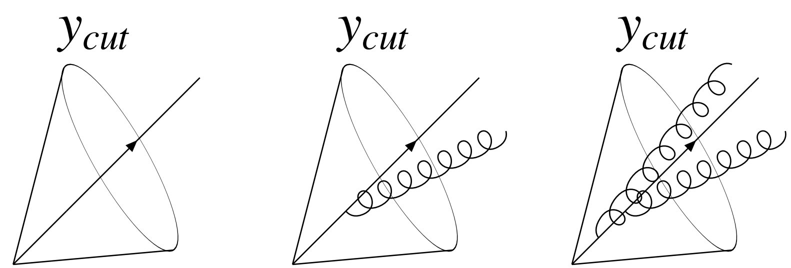
At leading order:



At next-to-leading order:



At next-to-next-to-leading order:



The master formula for the calculation of observables

$$\langle O \rangle = \underbrace{\int dx_1\, f(x_1) \int dx_2\, f(x_2)}_{\text{pdf's}} \underbrace{\frac{1}{2K(s)}}_{\text{flux factor}} \underbrace{\frac{1}{(2J_1+1)}\frac{1}{(2J_2+1)}\frac{1}{n_1 n_2}}_{\text{average over initial spins and colours}}$$

$$\times \sum_n \underbrace{\int d\phi_{n-2}}_{\text{integral over phase space}} \underbrace{O(p_1, ..., p_n)}_{\text{observable}} \underbrace{|\mathcal{A}_n|^2}_{\text{amplitude}}$$

Phase-space integration performed numerically by Monte-Carlo methods.

Observable infrared-safe:

$$\mathcal{O}_{n+1}(p_1, ..., p_{n+1}) \to \mathcal{O}_n(p'_1, ..., p'_n), \quad \text{(Single unresolved)}$$

$$\mathcal{O}_{n+2}(p_1, ..., p_{n+2}) \to \mathcal{O}_n(p'_1, ..., p'_n). \quad \text{(Double unresolved)}$$

Amplitudes $\mathcal{A}_n$ calculated in perturbation theory.

Calculation of observables

Perturbative expansion of the amplitude (LO, NLO, NNLO):

$$|\mathcal{A}_n|^2 = \underbrace{\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(0)}}_{\text{Born}} + \underbrace{\left(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)} + \mathcal{A}_n^{(1)*}\mathcal{A}_n^{(0)}\right)}_{\text{virtual}} + \underbrace{\left(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(2)} + \mathcal{A}_n^{(2)*}\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(1)*}\mathcal{A}_n^{(1)}\right)}_{\text{two-loop and loop-loop}},$$

$$|\mathcal{A}_{n+1}|^2 = \underbrace{\mathcal{A}_{n+1}^{(0)\,*}\mathcal{A}_{n+1}^{(0)}}_{\text{real}} + \underbrace{\left(\mathcal{A}_{n+1}^{(0)\,*}\mathcal{A}_{n+1}^{(1)} + \mathcal{A}_{n+1}^{(1)\,*}\mathcal{A}_{n+1}^{(0)}\right)}_{\text{loop+unresolved}},$$

$$|\mathcal{A}_{n+2}|^2 = \underbrace{\mathcal{A}_{n+2}^{(0)\,*}\mathcal{A}_{n+2}^{(0)}}_{\text{double unresolved}}.$$

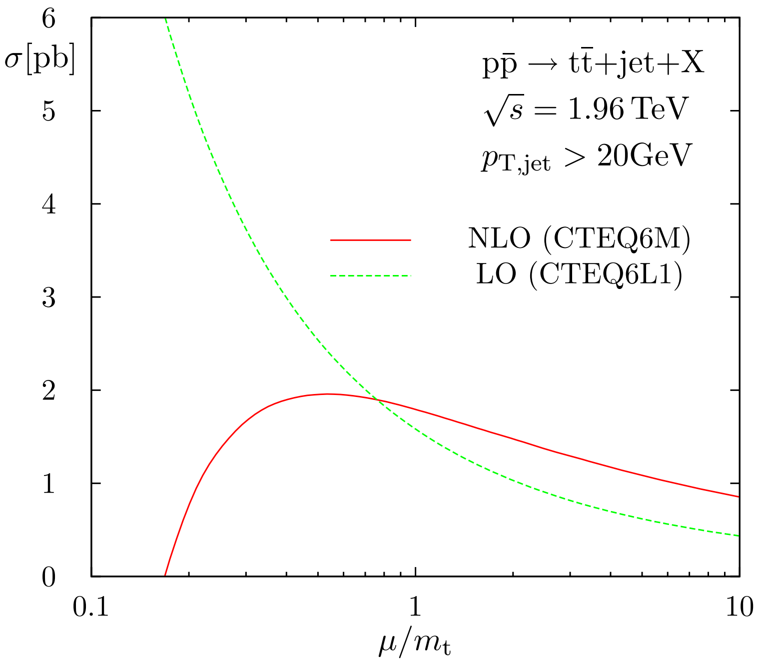
$\mathcal{A}_n^{(l)}$: amplitude with n external particles and l loops.

Dependence on renormalisation and factorisation scales

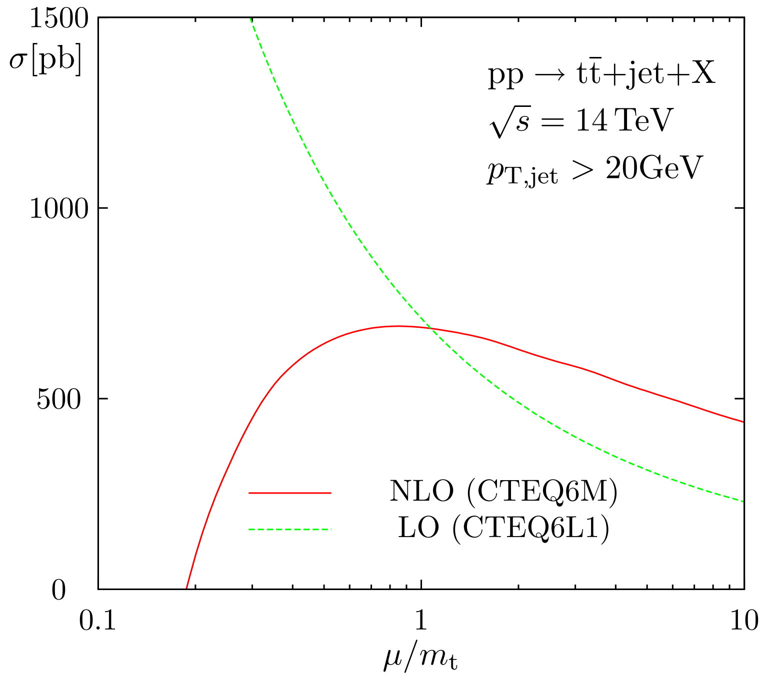
Example: $pp \longrightarrow t\bar{t} +$ jet.

Leading order is proportional to α_s^3 !

Tevatron:



LHC:



S. Dittmaier, P. Uwer and S.W.

Status

Economy: LO matrix elements automatically generated up to $2 \to 8$ or more.
Efficient integration over phase space

Madgraph/Madevent, Sherpa/Amegic++, Helac/Phegas, Comphep, Grace, Alpgen, ...

Quality segment: NLO calculations for many $2 \to 2$ and $2 \to 3$ processes, but no NLO calculations for $2 \to 4$ processes!

Electron-positron annihilation:

$e^+e^- \to$ 4 fermions: Denner, Dittmaier, Roth, Wieders

Automatisation in $2 \to 2,3$: Feynarts/Formcalc/Looptools, Grace, ...

Premium segment: NNLO calculations for a few selected $2 \to 1$ and $2 \to 2$ processes (Drell-Yan, W production, Higgs production), want standard $2 \to 2$ processes to a few percent accuracy.

Anastasiou, Melnikov, Petriello; Harlander, Kilgore; van Neerven, Ravindran, Smith;

Objectives

- Extract fundamental quantities like α_s to high precision
  ($e^+e^- \longrightarrow 3$ jets at LEP).

- Extract non-perturbative parameters (pdf's) to high precision
  (HERA).

- Predictions for multi-particle final states that occur at high rate and form background to new physics
  (NLO).

- Precise predictions for standard hard pp processes like W, Z, jets, top, Higgs
  (NNLO).

The NLO wish-list

The experimenter's wish list at Les Houches 2005:

	process	relevant for
1.	$pp \to VV +$ jet	$t\bar{t}H$, new physics
2.	$pp \to H +$ 2 jets	Higgs production by vector boson fusion
3.	$pp \to t\bar{t}b\bar{b}$	$t\bar{t}H$
4.	$pp \to t\bar{t} +$ 2 jets	$t\bar{t}H$
5.	$pp \to VVb\bar{b}$	$VBF \to H \to VV$, $t\bar{t}H$, new physics
6.	$pp \to VV +$ 2 jets	$VBF \to H \to VV$
7.	$pp \to V +$ 3 jets	various new physics signatures
8.	$pp \to VVV$	SUSY

$V \in \{Z, W, \gamma\}$.

Challenges

What are the bottle-necks ?

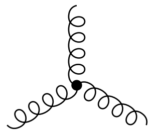
- Length: Perturbative calculations lead to expressions with a huge number of terms.
- Integrals: At one-loop and beyond, the occuring integrals cannot be simply looked up in an integral table.
- Divergences: At NLO and beyond, infrared divergences occur in intermediate stages, if massless particles are involved.

Brute force

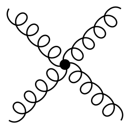
Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

2	4
3	25
4	220
5	2485
6	34300
7	559405
8	10525900

Feynman rules:



$$= g f^{abc}\left[(k_2-k_3)_\mu g_{\nu\lambda} + (k_3-k_1)_\nu g_{\lambda\mu} + (k_1-k_2)_\lambda g_{\mu\nu}\right]$$



$$= -ig^2\left[f^{abe}f^{ecd}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\right) + f^{ace}f^{ebd}\left(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\rho}g_{\lambda\nu}\right) + f^{ade}f^{ecb}\left(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho}\right)\right]$$

Feynman diagrams are not the method of choice !

Managing lengthy expressions

- Computer algebra
- Quantum number management
  - Colour decomposition
  - Spinor methods
  - Off-shell recurrence relations
  - Parke-Taylor formulae
- New developments: Twistor methods
  - MHV vertices
  - On-shell recursion relations

Computer algebra

Computer-intensive symbolic calculations in particle physics can be characterized by:

- Need for basic operations like addition, multiplication, sorting ...
- Specialized code usually written by the user
- No need for a system which knows “more” than the user!

CAS on the market:

- Commercial: Mathematica, Maple, Reduce, ...
- Non-commercial: FORM , GiNaC, ...

  Vermaseren; Bauer, Frink, Kreckel, Vollinga, ...

Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors carrying the colour structures multiplied by kinematic functions called partial amplitudes.

The partial amplitudes do not contain any colour information and are gauge-invariant. Each partial amplitude has a fixed cyclic order of the external legs.

Examples: The n-gluon amplitude:

$$\mathcal{A}_n(1,2,...,n) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \underbrace{2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)}_{\text{Chan Patton factors}} \underbrace{A_n\left(\sigma(1),...,\sigma(n)\right)}_{\text{partial amplitudes}}.$$

P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach,

F. A. Berends and W. Giele,

M. L. Mangano, S. J. Parke, and Z. Xu,

D. Kosower, B.-H. Lee, and V. P. Nair,

Z. Bern and D. A. Kosower.

The spinor helicity method

- Basic objects: Massless two-component Weyl spinors

$$|p\pm\rangle, \qquad \langle p\pm|$$

- Gluon polarization vectors:

$$\varepsilon_\mu^+(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

  q is an arbitrary light-like reference momentum. Dependency on q drops out in gauge invariant quantities.

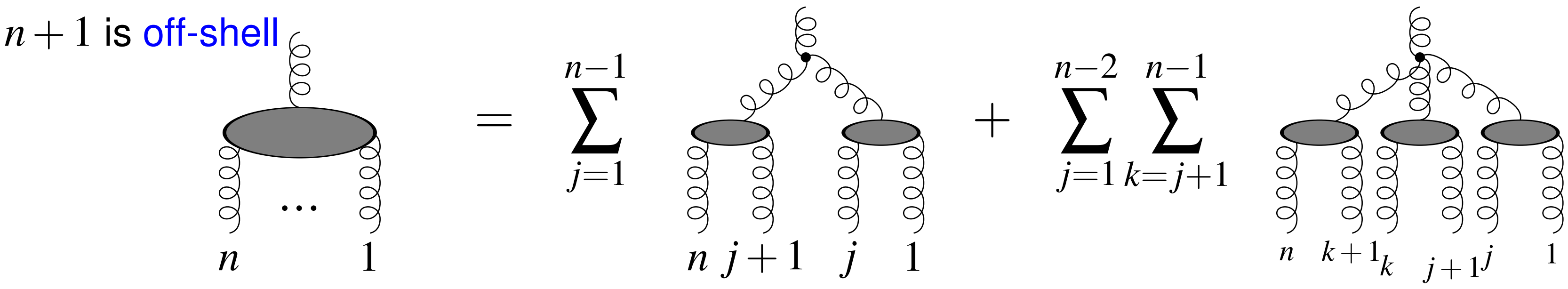
- A clever choice of the reference momentum can reduce significantly the number of diagrams which need to be calculated.

Berends, Kleiss, De Causmaecker, Gastmans and Wu; Xu, Zhang and Chang;

Kleiss and Stirling; Gunion and Kunszt

Recurrence relations

Off-shell currents provide an efficient way to calculate amplitudes:



Momentum conservation: $p_{n+1} = p_1 + p_2 + ... + p_n$.

On-shell condition for particles 1 to n: $p_j^2 = m_j^2$.

No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele,

D. A. Kosower.

The Parke-Taylor formulae

For specific helicity combinations the amplitudes have a remarkably simple analytic formula or vanish altogether:

$$\begin{aligned} A_n^{tree}(g_1^+,\ldots,g_n^+) &= 0, \\ A_n^{tree}(g_1^+,\ldots,g_j^-,\ldots,g_n^+) &= 0, \\ A_n^{tree}(g_1^+,\ldots,g_j^-,\ldots,g_k^-,\ldots,g_n^+) &= i\left(\sqrt{2}\right)^{n-2}\frac{\langle jk\rangle^4}{\langle 12\rangle\ldots\langle n1\rangle}. \end{aligned}$$

The n-gluon amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity is called a maximal-helicity violating amplitude (MHV amplitude).

F. A. Berends and W. T. Giele,

S. J. Parke and T. R. Taylor.

The CSW construction

Cachazo, Svrček and Witten proposed that the gluonic Born amplitude with an arbitrary helicity configuration can be calculated from diagrams with scalar propagators and new vertices, which are MHV-amplitudes continued off-shell.

$$A_n(1^+,...,j^-,...,k^-,...,n^+) \quad = \quad i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk\rangle^4}{\langle 12\rangle ... \langle n1\rangle}.$$

Off-shell continuation:

$$p^\flat \quad = \quad P - \frac{P^2}{2Pq} q.$$

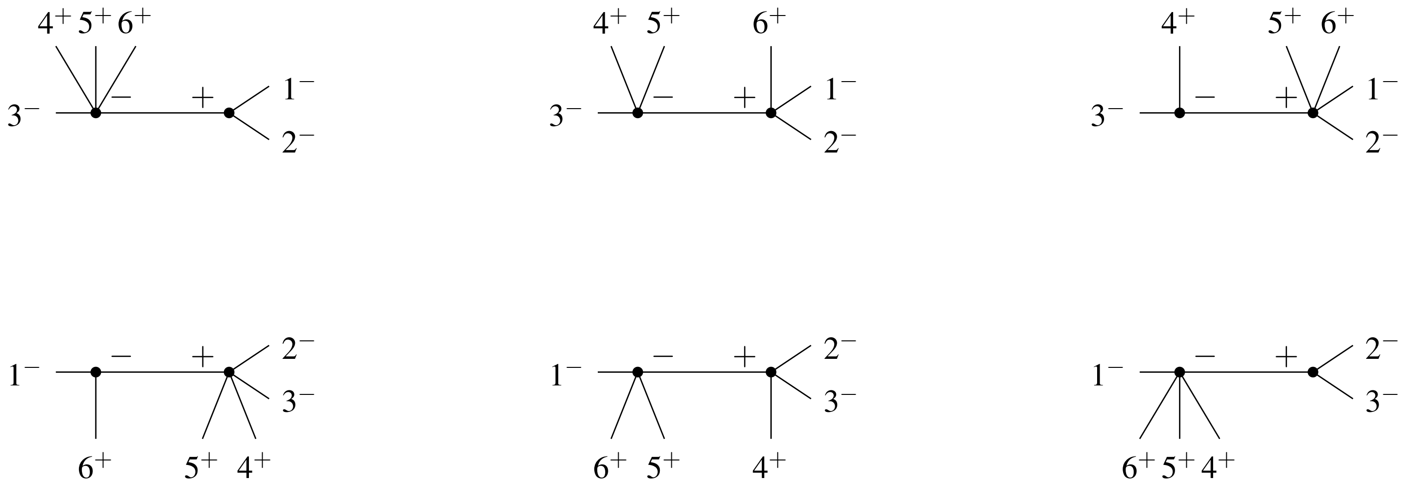
Propagators are scalars:

$$\frac{-i}{P^2}$$

Cachazo, Svrček and Witten, JHEP 0409:006, (hep-th/0403047)

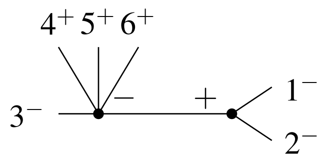
Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

Six diagrams, each consisting of two MHV-vertices:



Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

The first diagram yields:



$=$

$$\left[i\sqrt{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\left(-k_{12}^\flat\right)\rangle\langle\left(-k_{12}^\flat\right)1\rangle}\right] \frac{i}{k_{12}^2} \left[i\left(\sqrt{2}\right)^3 \frac{\langle 3k_{12}^\flat\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6k_{12}^\flat\rangle\langle k_{12}^\flat 3\rangle}\right]$$

Similar for the five other diagrams.

Compare this to

- a brute force approach (220 Feynman diagrams)

- colour-ordered amplitudes (36 diagrams)

The BCF recursion relations

R. Britto, F. Cachazo and B. Feng gave a recursion relation for the calculation of the n-gluon amplitude:

$$A_n\left(p_1, p_2, \ldots, p_{n-1}^-, p_n^+\right) =$$
$$\sum_{i=1}^{n-3} \sum_{\lambda=+,-} A_{i+2}\left(\hat{p}_n, p_1, p_2, \ldots, p_i, -\hat{P}_{n,i}^{\lambda}\right) \left(\frac{-i}{P_{n,i}^2}\right) A_{n-i}\left(\hat{P}_{n,i}^{-\lambda}, p_{i+1}, \ldots, p_{n-2}, \hat{p}_{n-1}\right).$$

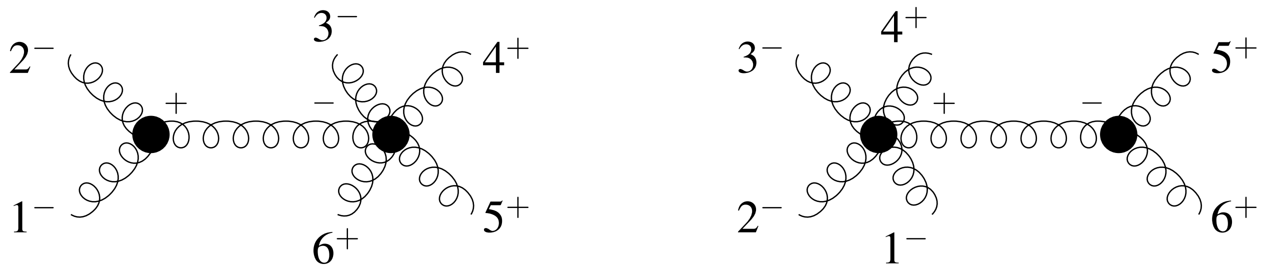
No off-shell continuation needed. The amplitudes on the r.h.s. are evaluated with shifted momenta.

Britto, Cachazo and Feng, Nucl. Phys. B715, (2005), 499, (hep-th/0412308);

Britto, Cachazo, Feng and Witten, Phys. Rev. Lett. 94:181602, (2005), (hep-th/0501052)

Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

Only two diagrams contribute:



$$A_6^{tree}(1^-,2^-,3^-,4^+,5^+,6^+) =$$

$$4i\left[\frac{\langle 6+|1+2|3+\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle s_{126}\langle 2+|1+6|5+\rangle} + \frac{\langle 4+|5+6|1+\rangle^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{156}\langle 2+|1+6|5+\rangle}\right]$$

The number of diagrams

Example: Number of diagrams contributing to the colour-ordered six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$:

brute force approach:	220
colour-ordered amplitudes:	36
MHV vertices:	6
on-shell recursion:	2

Calculating loop amplitudes

- Automated computation of one-loop amplitudes
  - Improvements of the Passarino-Veltman algorithm
  - Unitarity method
- Two-loop amplitudes and beyond
  - Mellin-Barnes transformation
  - Multiple polylogarithms
  - Sector decomposition

Automated NLO calculations

Automated NLO calculations for $2 \to n$ $(n = 4..6, 7)$ processes relevant for physics at the LHC and the ILC.

Technical challenges: Automated numerical evaluation of one-loop amplitudes.

A. Ferroglia, M. Passera, G. Passarino, and S. Uccirati,

Z. Nagy and D. E. Soper,

W. Giele, E. W. N. Glover, and G. Zanderighi,

F. del Aguila and R. Pittau,

T. Binoth, G. Heinrich, and N. Kauer,

A. Denner and S. Dittmaier,

A. van Hameren, J. Vollinga and S.W.

Reduction of tensor integrals

The Passarino-Veltman algorithm:

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{k_\mu k_\nu}{(k^2 - m_1^2)((k-p_1)^2 - m_2^2)((k-p_1-p_2)^2 - m_3^2)}$$
$$= \quad p_1^\mu p_1^\nu C_{21} + p_2^\mu p_2^\nu C_{22} + (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)\, C_{23} + g^{\mu\nu} C_{24}.$$

Inverting the linear system of equations introduces Gram determinants:

$$\Delta = \left| \begin{array}{cc} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{array} \right|.$$

Improved algorithms avoid these Gram determinants!

A. Denner and S. Dittmaier,

T. Binoth, G. Heinrich, and N. Kauer.

Reduction of tensor integrals

Loop momentum in the numerator is always contracted into an external structure:

$$\langle a_1 - |\gamma_{\mu_1}| b_1 - \rangle \dots \langle a_r - |\gamma_{\mu_r}| b_r - \rangle \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{k^{\mu_1} \dots k^{\mu_r}}{k^2 (k-p_1)^2 \dots (k - p_1 - \dots p_{n-1})^2}.$$

Use spinor methods to decompose k^μ into

$$k^\mu \quad = \quad c_1\, l_1^\mu + c_2\, l_2^\mu + c_3\, \langle l_1 - |\gamma^\mu| l_2 - \rangle + c_4\, \langle l_2 - |\gamma^\mu| l_1 - \rangle.$$

F. del Aguila and R. Pittau,

A. van Hameren, J. Vollinga and S.W.

Reduction of scalar integrals

One-loop integrals with more than four propagators can always be reduced to integrals with maximally four propagators.

Melrose (1965)

Basic idea: In a space of dimension four there can be no more than four linear independet vectors.

The proof can be extended towards integrals computed within dimensional regularization.

Reduction of scalar integrals

Reduction of pentagons (W. van Neerven and J. Vermaseren; Z. Bern, L. Dixon, and D. Kosower):

$$I_5 = \sum_{i=1}^{5} b_i I_4^{(i)} + \mathcal{O}(\varepsilon).$$

Reduction of hexagons (T. Binoth, J. P. Guillet, and G. Heinrich):

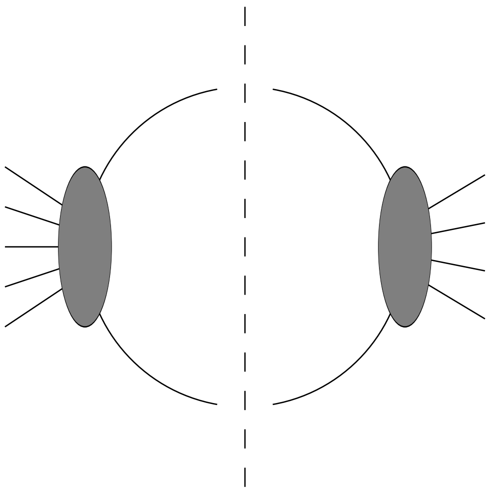
$$I_6 = \sum_{i=1}^{6} b_i I_5^{(i)}.$$

Reduction of scalar integrals with more than six propagators (G. Duplancic and B. Nizic):

$$I_n = \sum_{i=1}^{n} r_i I_{n-1}^{(i)}.$$

Here, the decomposition is no longer unique.

Unitarity method



$$A^{1-loop} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2 + i\varepsilon} \frac{1}{k_2^2 + i\varepsilon} A_L^{tree} A_R^{tree}$$

$+$ cut free pieces

The cut-construction simplifies the calculation of one-loop amplitudes, as cancellations occur already inside A_L^{tree} and A_R^{tree}.

Theorem: One-loop amplitudes in massless supersymmetric gauge theories with no superpotential can be completely constructed from their cuts.

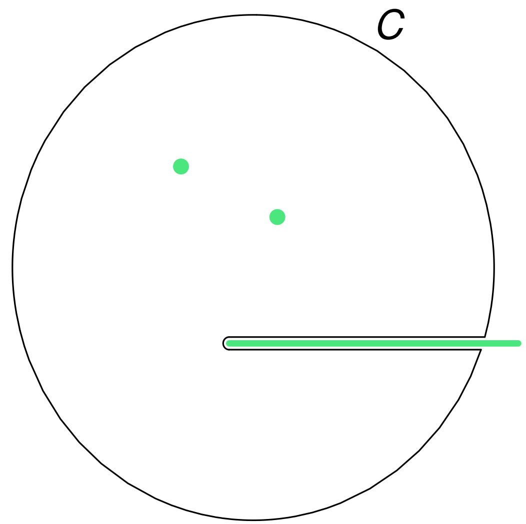
Bern, Dixon, Dunbar and Kosower

Loop amplitudes

Split QCD amplitudes into $N = 4$ and $N = 1$ SUSY pieces and a scalar part.

Loop amplitudes have branch cuts:
Get branch cuts from the unitarity method.
Use recursion relations for the rational pieces.

$$A_n(0) \quad = \quad C_\infty - \sum_{poles} \text{res}\, \frac{A_n(z)}{z} - \int_{B_0}^{\infty} \frac{dz}{z} \text{Disc}\, A_n(z)$$



Complications: Boundary terms, double poles.

Brandhuber, Spence and Travaglini;

Bern, Dixon, Kosower

One-loop corrections $A_n^{1-loop}(1^-, 2^-, 3^+, \ldots, n^+)$ to adjacent MHV amplitudes have been calculated.

Forde, Kosower

The one-loop six-gluon amplitude

Analytic computation:

Bedford, Berger, bern Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu.

$$\mathcal{A}_n = \mathcal{A}_n^{\mathcal{N}=4} - 4\mathcal{A}_n^{\mathcal{N}=1} + \mathcal{A}_n^{\mathcal{N}=0}$$

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0$ (cut)	$\mathcal{N}=0$ (rat)
$--++++$	BDDK (94)	BDDK (94)	BDDK (94)	BDK (94)
$-+-+++$	BDDK (94)	BDDK (94)	BBST (04)	BBDFK (06), XYZ (06)
$-++-++$	BDDK (94)	BDDK (94)	BBST (04)	BBDFK (06), XYZ (06)
$---+++$	BDDK (94)	BDDK (94)	BBDI (05), BFM (06)	BBDFK (06), XYZ (06)
$--+-++$	BDDK (94)	BBDP (05), BBCF (05)	BFM (06)	XYZ (06)
$-+-+-+$	BDDK (94)	BBDP (05), BBCF (05)	BFM (06)	XYZ (06)

Numerical check:

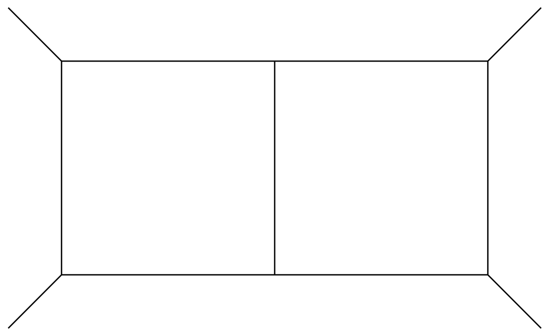
Ellis, Giele, Zanderighi (2006)

The calculation of two-loop integrals

- Techniques to calculate two-loop integrals
  - Mellin-Barnes transformation, Smirnov ’99, Tausk ’99.
  - Differential equations, Gehrmann, Remiddi ‘00.
  - Nested sums, Moch, Uwer, S.W. ‘01.
  - Sector decomposition (numerical), Binoth, Heinrich, ’00.
- Methods to reduce the work-load:
  - Integration-by-parts, Chetyrkin, Kataev, Tkachov ‘81.
  - Reduction algorithms, Tarasov ‘96, Laporta ’01.
  - Cut technique Bern, Dixon, Kosower, ’00

The double-box integral

Two-loop amplitudes for $2 \to 2$ processes involve the double-box integral:



- First calculated by Smirnov in 1999.
- Calculation based on Mellin-Barnes representation.
- Result expressed in harmonic polylogarithms.

Mellin-Barnes

Mellin-Barnes transformation:

$$(A_1 + A_2 + \ldots + A_n)^{-c} = \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int\limits_{-i\infty}^{i\infty} d\sigma_1 \ldots \int\limits_{-i\infty}^{i\infty} d\sigma_{n-1}$$

$$\times \Gamma(-\sigma_1) \ldots \Gamma(-\sigma_{n-1}) \Gamma(\sigma_1 + \ldots + \sigma_{n-1} + c) \, A_1^{\sigma_1} \ldots A_{n-1}^{\sigma_{n-1}} A_n^{-\sigma_1 - \ldots - \sigma_{n-1} - c}$$

The contour is such that the poles of $\Gamma(-\sigma)$ are to the right and the poles of $\Gamma(\sigma + c)$ are to the left.

Converts a sum into products and is therefore the "inverse" of Feynman parametrization.

Smirnov; Tausk; Davydychev; Bierenbaum, S.W.; Czakon; Anastasiou, Daleo; Gluza, Kajda, Riemann;

Multiple polylogarithms

- Definition:

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = \sum_{i_1>i_2>...>i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

(Goncharov; Borwein, Bradley, Broadhurst and Lisonek)

- Special subsets: Harmonic polylogs, Nielsen polylogs, classical polylogs

  (Remiddi and Vermaseren, Gehrmann and Remiddi).

- Have also an integral representation.

- Obey two Hopf algebras (Moch, Uwer, S.W.).

- Can be evaluated numerically for all complex values of the arguments

  (Gehrmann and Remiddi, Vollinga and S.W.).

Harmonic sums

- Definition:

$$S_{m_1,\ldots,m_k}(N) = \sum_{i_1 \ge i_2 \ge \ldots \ge i_k \ge 1}^{N} \frac{1}{i_1^{m_1}} \frac{1}{i_2^{m_2}} \cdots \frac{1}{i_k^{m_k}}.$$

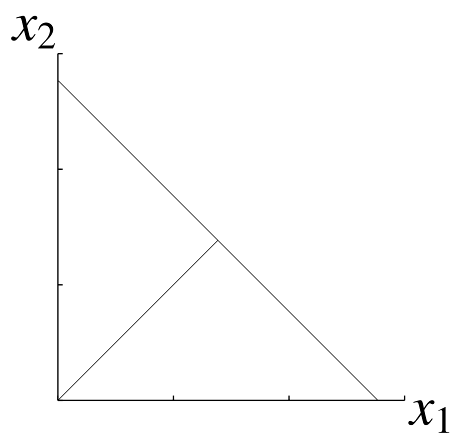
Gonzale-Arroyo, Lopez, Yndurain ’79, Vermaseren ’98, Blümlein, Kurth ’98

- Obey a Hopf algebra Vermaseren ’98, Moch, Uwer, S.W. ’01
- Mellin transforms of harmonic polylogarithms Vermaseren ’98
- Automated manipulations by computer algebra programs summer, nestedsums, xsummer, ...

Sector decomposition

Disentangle overlapping divergences:

$$\int_0^1 \frac{dx_1}{x_1^{1-\varepsilon}} \int_0^{1-x_1} \frac{dx_2}{x_2^{1-\varepsilon}} \frac{1}{x_1+x_2}$$



Decompose into several subsectors:

Sector $x_1 > x_2$: $\quad x_2' = x_2/x_1$

Sector $x_1 < x_2$: $\quad x_1' = x_1/x_2$

Can be applied to loop integrals and phase space integrals !

Roth, Denner; Binoth, Heinrich;

The calculation of two-loop amplitudes

- Calculation of two-loop amplitudes
  - Bhabha, Bern, Dixon, Ghinculov ‘01.
  - $pp \longrightarrow$ 2 jets, Anastasiou, Glover, Oleari, Tejeda-Yeomans ’01;
    Bern, De Freitas, Dixon, Ghinculov, Wong ’01.
  - $e^+e^- \longrightarrow$ 3 jets, L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis and E. Remiddi ’02;
    S. Moch, P. Uwer and S.W. ’02
  - Higgs production, Harlander, Kilgore; Catani, de Florian, Grazzini; Anastasiou, Melnikov;
  - Drell-Yan, Anastasiou, Dixon, Melnikov, Petriello; Ravindran, Smith, van Neerven

- Calculation of three-loop splitting functions S. Moch, J. Vermaseren and A. Vogt ’04;

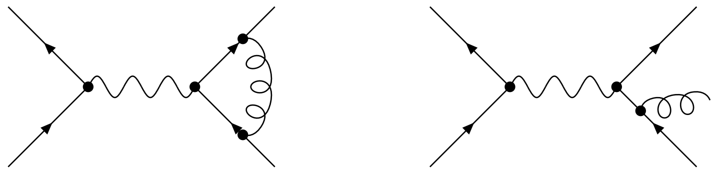
Cancellation of divergences

- Infrared divergences at NLO
  - General methods
  - Automatisation
- Infrared divergences at NNLO
  - Status

Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, loop integrals can have infrared divergences.

For each IR divergence there is a corresponding divergence with the opposite sign in the real emission amplitude, when particles becomes soft or collinear (e.g. unresolved).



The Kinoshita-Lee-Nauenberg theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

- Phase space slicing
  - e^+e^-: W. Giele and N. Glover, (1992)
  - initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
  - massive partons, fragmentation: S. Keller and E. Laenen, (1999)
- Subtraction method
  - residue approach: S. Frixione, Z. Kunzst and A. Signer, (1995)
  - dipole formalism: S. Catani and M. Seymour, (1996)
  - massive partons: L. Phaf and S.W. (2001), S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V$$

$$= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

The subtraction terms

The approximation term $d\sigma^A$ is given as a sum over dipoles:

$$d\sigma^A = \sum_{pairs\, i,j} \sum_{k \neq i,j} \mathcal{D}_{ij,k}.$$

Each dipole contribution has the following form:

$$\mathcal{D}_{ij,k} = -\frac{1}{2p_i \cdot p_j} \mathcal{A}_n^{(0)\,*}\left(p_1, \ldots, \tilde{p}_{(ij)}, \ldots, \tilde{p}_k, \ldots\right) \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k} \mathcal{A}_n^{(0)}\left(p_1, \ldots, \tilde{p}_{(ij)}, \ldots, \tilde{p}_k, \ldots\right).$$

- Colour correlations through $\mathbf{T}_k \cdot \mathbf{T}_{ij}$.
- Spin correlations through $V_{ij,k}$.

The physical origin of the correlations

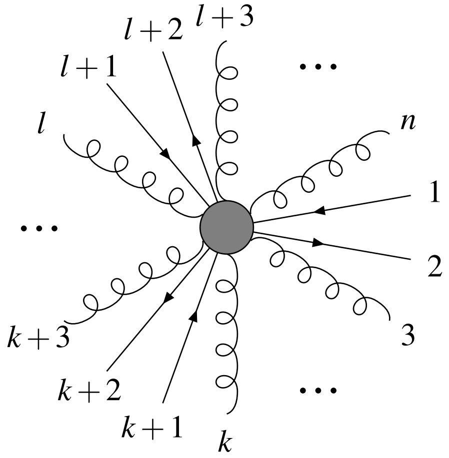
- In the soft limit, amplitudes factorize completely in spin space, but colour correlations remain.

- In the collinear limit, amplitudes factorize completely in colour space, but spin correlations remain.

Solution: Use colour decomposition and calculate helicity amplitudes.

Colour structures and cyclic ordering

For Born graphs: cyclic order such that a quark follows immediately its corresponding antiquark in the clockwise orientation.
Treat a $(\bar{q}, q)$ pair as a pseudo-leg.
All possible cyclic orderings are obtained by summing over all permutations of the pseudo-legs and factoring out the cyclic permutations.



Colour cluster: part of an amplitude, which is connected to the rest of the amplitude only by an $U(1)$ gluon and which does not contain by itself any $U(1)$ gluon.

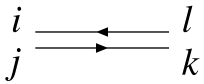
S.W. '05

The double line notation

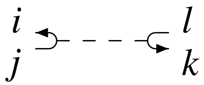
Replace a colour index in the adjoint representation by two indices in the fundamental representation:

$$V^a E^a \quad = \quad = \left(\sqrt{2} T^a_{ij} V^a\right) \left(\sqrt{2} T^b_{ji} E^b\right).$$

Then split a $SU(N)$ gluon into an $U(N)$-part and an $U(1)$-part:

$U(N):$ 

$= \quad \delta_{il}\delta_{kj},$

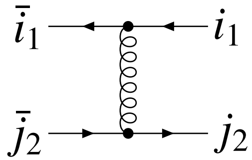
$U(1):$ 

$= \quad -\frac{1}{N}\delta_{ij}\delta_{kl}.$

One can show that the $U(1)$ gluon couples only to quarks.

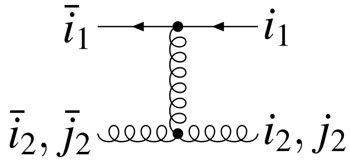
Colour correlations

Quark-antiquark:



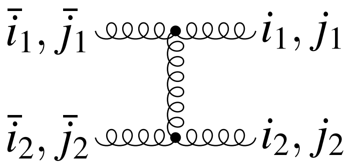
$$= \quad -\frac{1}{2}\left(\delta_{\bar{i}_1 \bar{j}_2}\delta_{j_2 i_1} - \frac{1}{N}\delta_{\bar{i}_1 i_1}\delta_{j_2 \bar{j}_2}\right)$$

Quark-gluon:



$$= \quad \frac{1}{2}\left(\delta_{\bar{i}_1 i_2}\delta_{\bar{i}_2 i_1}\delta_{j_2 \bar{j}_2} - \delta_{\bar{i}_1 \bar{j}_2}\delta_{j_2 i_1}\delta_{\bar{i}_2 i_2}\right)$$

Gluon-gluon:



$$= \quad \frac{1}{2}\left(\delta_{\bar{i}_1 i_1}\delta_{\bar{i}_2 i_2}\delta_{j_1 \bar{j}_2}\delta_{j_2 \bar{j}_1} - \delta_{\bar{i}_1 i_1}\delta_{j_2 \bar{j}_2}\delta_{j_1 i_2}\delta_{\bar{i}_2 \bar{j}_1} - \delta_{j_1 \bar{j}_1}\delta_{\bar{i}_2 i_2}\delta_{\bar{i}_1 \bar{j}_2}\delta_{j_2 i_1} + \delta_{j_1 \bar{j}_1}\delta_{j_2 \bar{j}_2}\delta_{\bar{i}_1 i_2}\delta_{\bar{i}_2 i_1}\right)$$

The subtraction method at NNLO

- Singular behaviour
  - Factorization of tree amplitudes in double unresolved limits, Berends, Giele, Cambell, Glover, Catani, Grazzini, Del Duca, Frizzo, Maltoni, Kosower '99
  - Factorization of one-loop amplitudes in single unresolved limits, Bern, Del Duca, Kilgore, Schmidt, Kosower, Uwer, Catani, Grazzini, '99
- Extension of the subtraction method to NNLO Kosower; S.W.; Kilgore; Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Frixione, Grazzini; Somogyi, Trócsányi and Del Duca;
- Cancellation based on sector decomposition Anastasiou, Melnikov, Petriello; Heinrich;
- Applications:
  - $pp \to W$, Anastasiou, Dixon, Melnikov, Petriello '03,
  - $pp \to H \to \gamma\gamma$, Anastasiou, Dixon, Melnikov, Petriello '05,
  - $e^+e^- \to$ 2 jets, Anastasiou, Melnikov, Petriello '04, S.W. '06

Summary

- Length:
  - Computer algebra
  - Quantum number management: Colour decomposition, spinor methods
  - New developments: Twistor methods

- Integrals:
  - Automated computation of one-loop amplitudes: Improvements of the Passarino-Veltman algorithm, unitarity method
  - Two-loop amplitudes and beyond: Mellin-Barnes, multiple polylogarithms, sector decomposition

- Divergences:
  - Automated cancellation of infrared divergences at NLO
  - R & D at NNLO