# NNLO QCD calculation for $e^+e^- \to 3$ jets

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# Perturbative NNLO calculations

The experimental precision has to be matched with an improvement in the accuracy of theoretical predictions.

Fully differential NNLO programs for

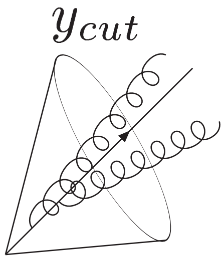
- Bhabha scattering
- $pp \to 2$ jets
- $e^+e^- \to 3$ jets

which allow the calculation of any infrared safe observable.

# $e^+e^- \to 3$ jets

- Measurement of $\alpha_s$ using data of $e^+e^- \to 3$ jets. A NNLO calculation is expected to reduce the theoretical error in the extraction of $\alpha_s$ down to 1%.

- A NNLO calculation models the jet structure more accurately and should allow to improve the knowledge on the interplay between perturbative QCD and power corrections.



# Recent progress in NNLO calculations

- Two loop master integrals: Double boxes, crossed double boxes, pentaboxes
  - Mellin-Barnes: V.A. Smirnov, J.B. Tausk
  - Differential equations: T. Gehrmann and E. Remiddi
  - Negative dimensions: C. Anastasiou, N. Glover and C. Oleari
- Two loop amplitudes:
  - Bhabha: Z. Bern, L. Dixon and A. Ghinculov
  - $pp \to 2$ jets: C. Anastasiou, N. Glover, C. Oleari and M. Tejeda-Yeomans
- Singular limits:
  - Double unresolved: J. Campbell and N. Glover, S. Catani and M. Grazzini
  - One-loop + unresolved: Z. Bern, V. Del Duca, W. Kilgore and C. Schmidt, D. Kosower and P. Uwer
  - Pole structure of two-loop: S. Catani
- Numerical cross-checks:
  - Sector decomposition: Th. Binoth and G. Heinrich
  - Difference equations: S. Laporta

# To-Do list

- Calculation of the two-loop amplitudes for $e^+e^- \to 3$ jets.

  Requires: Tensor reduction. The result will depend on two ratios of scales, for example $s_{12}/s_{123}$ and $s_{23}/s_{123}$.
  The calculated two-loop amplitudes for Bhabha scattering and $pp \to 2$ jets depend only on one ratio $x = s/t$.

- Cancellation of IR divergences has to be done before any Monte Carlo integration.

  Requires: Extension of the subtraction or slicing method to NNLO. Analytic integration over unresolved regions or dipole phase space.

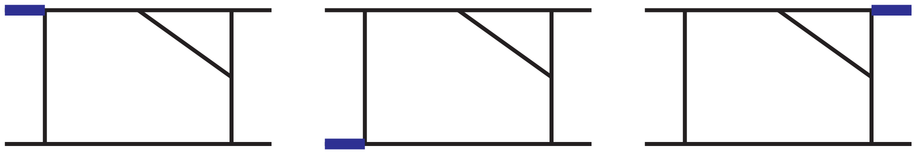
- The final numerical computer program.

  Requires: Stable and efficient numerical methods.

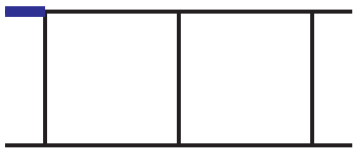
# Two-loop diagrams
# and how we break them into little pieces

## Basic topologies at two loops

- penta-box



- double-box



- nonplanar-box

# Tensor reduction

The standard approach:

- Using Schwinger parameters a tensor integral can be related to a scalar integral with higher powers of the propagators and different values of $D$ (Tarasov '96).
- Integration-by-parts, Lorentz-invariance identities and form-factor relations can be used to reduce the powers of the propagators and to relate master integrals in higher dimensions to master integrals in $D = 4 - 2\varepsilon$ dimensions.
- Complicated diagrams mapped onto simpler ones.
- This has to be done for each topology.

# Reduction identities

- Integration by parts

  'it Hooft, Veltman '72; Chetyrkin, Tkachov '81

$$\int d^D l_1 d^D l_2 \, \frac{\partial}{\partial l_i^\mu} \left[ l_j^\mu \, f(l_1^2, ... l_n^2) \right] = 0$$

- Lorentz invariance relations

  Gehrmann, Remiddi '99

$$\epsilon_{\mu\nu} \left( p_1^\mu \frac{\partial}{\partial p_1^\nu} + p_2^\mu \frac{\partial}{\partial p_2^\nu} + p_3^\mu \frac{\partial}{\partial p_3^\nu} \right) \int d^D l_1 d^D l_2 \, f(l_1^2, \ldots l_n^2) = 0$$

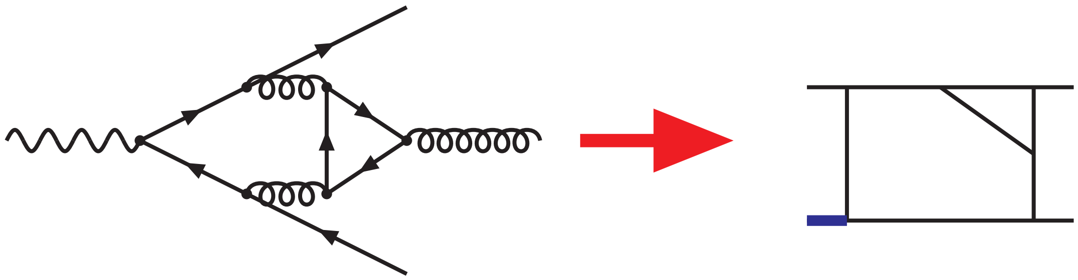
- Passarino–Veltman form factors

  Larin '81; Moch, Vermaseren '99

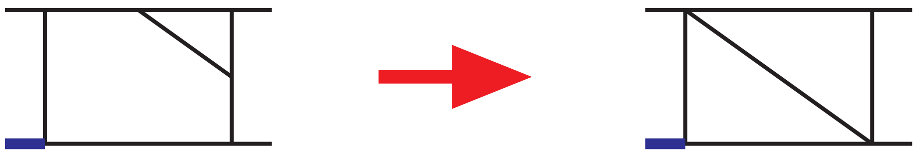
$$\frac{\partial}{\partial p_i^\mu} \int d^D l_1 d^D l_2 \, \left[ l_j^\mu \, f(l_1^2, \ldots l_n^2) \right] = \frac{\partial}{\partial p_i^\mu} (p_1^\mu I_1 + p_2^\mu I_2 + p_3^\mu I_3)$$

## Simple example

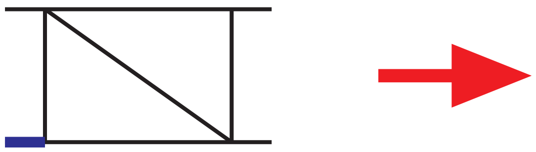
- Transform a diagram into a basic topology



- Reduction of the penta-box to the C-topology



- Evaluation of the C-topology



multiple polylogs

# The two-loop C-topology

The result for the C-topology in arbitrary dimensions and with arbitrary powers of the propagators:

$$
\frac{\Gamma(2m-2\varepsilon-\nu_{1235})\Gamma(1+\nu_{1235}-2m+2\varepsilon)\Gamma(2m-2\varepsilon-\nu_{2345})\Gamma(1+\nu_{2345}-2m+2\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu_5)\Gamma(3m-3\varepsilon-\nu_{12345})}
$$

$$
\cdot\frac{\Gamma(m-\varepsilon-\nu_5)\Gamma(m-\varepsilon-\nu_{23})}{\Gamma(2m-2\varepsilon-\nu_{235})}(-s_{123})^{2m-2\varepsilon-\nu_{12345}}\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\frac{x_1^{i_1}}{i_1!}\frac{x_2^{i_2}}{i_2!}
$$

$$
\cdot\left[\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+\nu_2)\Gamma(i_1+i_2-2m+2\varepsilon+\nu_{12345})\Gamma(i_1+i_2-m+\varepsilon+\nu_{235})}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+\nu_{23})}\right.
$$

$$
-x_1^{2m-2\varepsilon-\nu_{1235}}
$$

$$
\cdot\frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+\nu_2)\Gamma(i_1+i_2+\nu_4)\Gamma(i_1+i_2+m-\varepsilon-\nu_1)}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{15})}
$$

$$
-x_2^{2m-2\varepsilon-\nu_{2345}}
$$

$$
\cdot\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+2m-2\varepsilon-\nu_{345})\Gamma(i_1+i_2+\nu_1)\Gamma(i_1+i_2+m-\varepsilon-\nu_4)}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{45})}
$$

$$
+x_1^{2m-2\varepsilon-\nu_{1235}}x_2^{2m-2\varepsilon-\nu_{2345}}
$$

$$
\cdot\frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+2m-2\varepsilon-\nu_{345})}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})}
$$

$$
\left.\cdot\frac{\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{235})\Gamma(i_1+i_2+3m-3\varepsilon-\nu_{12345})}{\Gamma(i_1+i_2+4m-4\varepsilon-\nu_{12345}-\nu_5)}\right],
$$

with $D = 2m-2\varepsilon$, $x_1 = (-s_{12})/(-s_{123})$ and $x_2 = (-s_{23})/(-s_{123})$.

This sum can be expanded systematically in $\varepsilon$.

# Introducing nested sums

- Definition of nested sums (Z-sums)

$$Z(n; m_1, ..., m_k; x_1, ..., x_k) =$$
$$= \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} Z(i-1; m_2, ..., m_k; x_2, ..., x_k),$$
$$Z(n) = 1 \quad \text{for } n \geq 0.$$

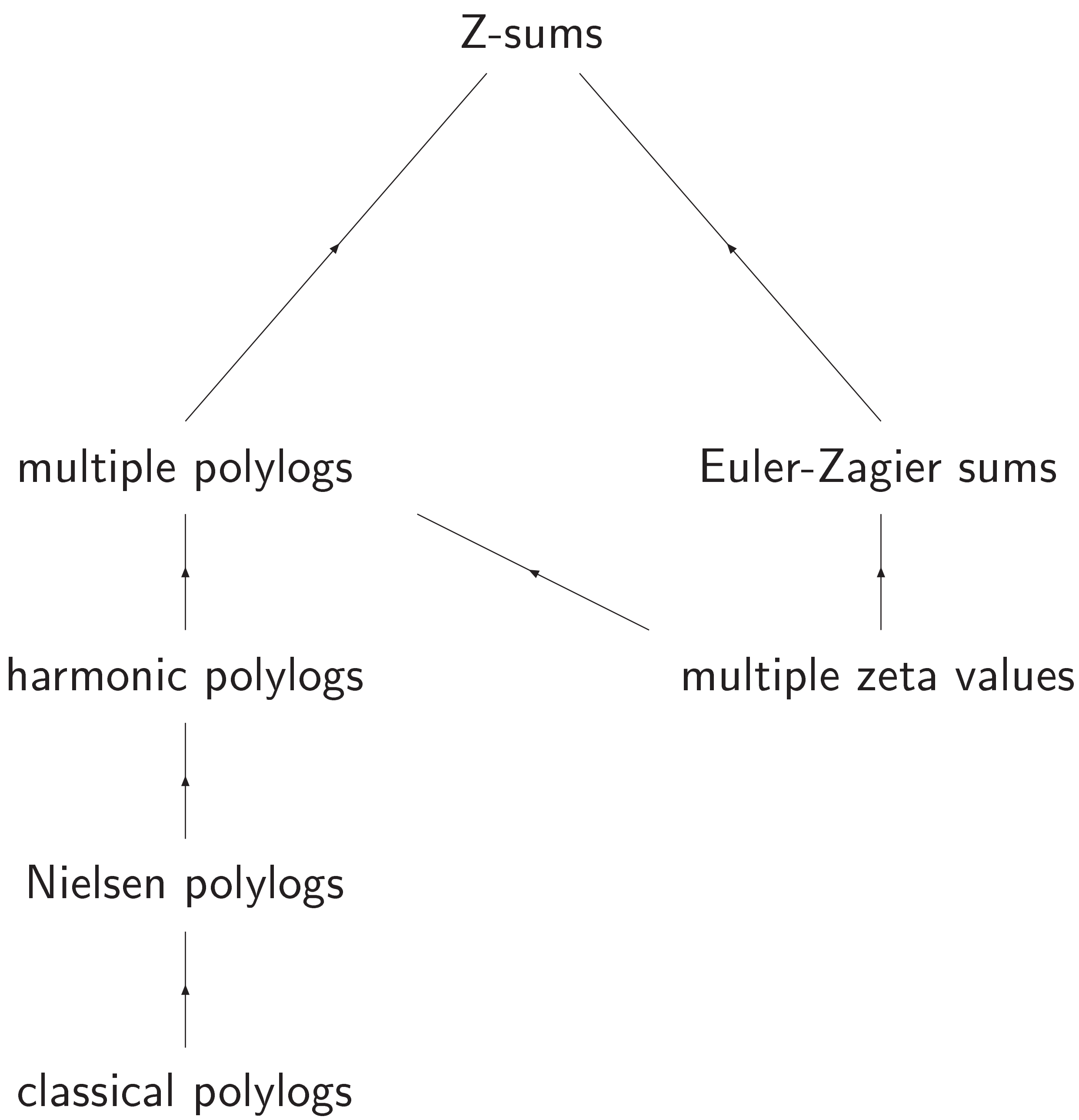
- Multiple polylogarithms are a special subset

- Euler-Zagier sums are a special subset

- The nested sums form a Hopf algebra

Z-sums interpolate between multiple polylogarithms and Euler-Zagier sums.

# Inheritance



## Hopf algebras and renormalization of quantum field theories

Short-distance singularities of the perturbative expansion of quantum field theories require renormalization (Bogoliubov, Parasuik, Hepp, Zimmermann).

The combinatorics involved in the renormalization is governed by a Hopf algebra (Kreimer, Connes).

The model for this Hopf algebra is the Hopf algebra of rooted trees.

Also the Z-sums are described by the Hopf algebra of rooted trees.

# Subtraction method

The NNLO cross section receives contributions from

$$
\begin{aligned}
d\sigma_{n+2}^{(0)} &= \left( \mathcal{A}_{n+2}^{tree\,*} \, \mathcal{A}_{n+2}^{tree} \right) \theta_{n+2} d\phi_{n+2}, \\
d\sigma_{n+1}^{(1)} &= \left( \mathcal{A}_{n+1}^{tree\,*} \, \mathcal{A}_{n+1}^{1-loop} + \mathcal{A}_{n+1}^{1-loop\,*} \, \mathcal{A}_{n+1}^{tree} \right) \theta_{n+1} d\phi_{n+1}, \\
d\sigma_{n}^{(2)} &= \left( \mathcal{A}_{n}^{tree\,*} \, \mathcal{A}_{n}^{2-loop} + \mathcal{A}_{n}^{2-loop\,*} \, \mathcal{A}_{n}^{tree} + \mathcal{A}_{n}^{1-loop\,*} \, \mathcal{A}_{n}^{1-loop} \right) \theta_{n} d\phi_{n},
\end{aligned}
$$

To obtain the NLO cross section for $(n+1)$ partons (Catani and Seymour '97):

$$
\int d\sigma_{n+2}^{(0)} + \int d\sigma_{n+1}^{(1)} = \int \left( d\sigma_{n+2}^{(0)} - d\sigma_{n+1}^{A} \right) + \int \left( d\sigma_{n+1}^{(1)} + d\sigma_{n+1}^{A} \right).
$$

Similar, we rewrite the NNLO cross section as

$$
\begin{aligned}
\int d\sigma^{NNLO} &= \int \left( d\sigma_{n+2}^{(0)} - d\sigma_{n+1}^{A} - d\sigma_{n}^{A} + d\sigma_{n}^{AA} \right) \\
&+ \int \left( d\sigma_{n+1}^{(1)} + d\sigma_{n+1}^{A} - d\sigma_{n}^{A,loop} - d\sigma_{n}^{AA,loop} \right) \\
&+ \int \left( d\sigma_{n}^{(2)} + d\sigma_{n}^{A,loop} + d\sigma_{n}^{AA,loop} + d\sigma_{n}^{A} - d\sigma_{n}^{AA} \right).
\end{aligned}
$$

# Subtraction method for one-loop amplitudes

- Decompose amplitudes into primitive amplitudes (fixed cyclic ordering and definitive routing of fermion lines).
- From the known soft and collinear properties obtain the subtraction terms.
- Integrate over unresolved phase space using the technique of nested sums (L. Phaf and S.W., '01)

Soft radiation pattern more complicated: Simple dipole picture with emitter and spectator not sufficient !

Subtraction method requires a mapping between the $(n+2), (n+1)$ parton phase space and the $n$ parton phase space. This mapping can be inverted and used to provide a more stable method for numerical integration (S.W. and D. Kosower '99).

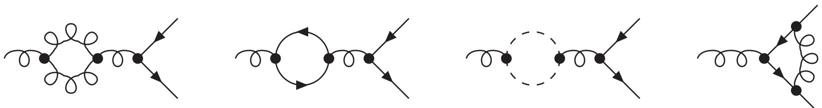
# Example : The splitting $g \to \bar{q}q$

## The general structure:

$$\mathcal{A}_{n+1}^{tree\,*}\,\mathcal{A}_{n+1}^{1-loop} + \mathcal{A}_{n+1}^{1-loop\,*}\,\mathcal{A}_{n+1}^{tree} \to$$

$$\text{Sing}^{tree}\left(\mathcal{A}_{n}^{tree\,*}\,\mathcal{A}_{n}^{1-loop} + \mathcal{A}_{n}^{1-loop\,*}\,\mathcal{A}_{n}^{tree}\right) + \text{Sing}^{1-loop}\,\mathcal{A}_{n}^{tree\,*}\,\mathcal{A}_{n}^{tree}$$

## Decompose loops into primitive parts:




## A subtraction term for each primitive structure:

$$\langle\mu|V_{\bar{q}_i q_j,k}^{1-loop,RF,[1]}|\nu\rangle =$$

$$\left[-g^{\mu\nu} - \frac{4}{2p_ip_j}S^{\mu\nu}\right]\cdot c_\Gamma\left(\frac{\mu^2}{-2p_ip_j}\right)^{\varepsilon}\left\{\frac{1}{\varepsilon^2} + \frac{3}{2}\frac{1}{\varepsilon(1-2\varepsilon)} + \frac{1}{1-2\varepsilon}\right\}$$

## The integrated counter-part:

$$\int d\phi_{dipole}\frac{1}{2p_ip_j}V_{\bar{q}_i q_j,k}^{1-loop,RF,[1]} = \left(-g^{\mu\nu}\right)c_\Gamma^2\left(\frac{\mu^2}{P^2}\right)^{\varepsilon}\left(\frac{\mu^2}{-P^2}\right)^{\varepsilon}$$

$$\cdot\left\{-\frac{1}{3}\frac{1}{\varepsilon^3} - \left(\frac{23}{18}+\frac{1}{3}\gamma\right)\frac{1}{\varepsilon^2} - \left(\frac{1}{6}\gamma^2+\frac{23}{18}\gamma+\frac{118}{27}\right)\frac{1}{\varepsilon}\right.$$

$$\left.-\left(\frac{7}{9}\zeta_3+\frac{1}{18}\gamma^3+\frac{23}{36}\gamma^2+\frac{118}{27}\gamma+\frac{1}{6}\pi^2+\frac{4075}{324}\right)\right\} + O(\varepsilon)$$

# Implementation on a computer

Requirements on a computer algebra system:

- Efficiency in performance: Compilation
- Object-oriented, strong type checking
- Short development cycles: Interactive use and high-quality output
- Source code freely available

The GiNaC library allows symbolic calculation in C++ (Ch.Bauer, A.Frink and R.Kreckel).

gTybalt for interactive use (R. Marani and S.W.) .

`http://www.fis.unipr.it/~stefanw/gtybalt.html`

# Project management

- Innovation and spin-offs:
  - Expansion of higher transcedental functions based on the Hopf algebra of nested sums.
  - Free computer algebra system
- Risk hedging: Conservative and innovative approaches
  - Integration-by-parts vs. nested sums.
  - FORM vs. C++
- Synergy effects:
  - Nested sums used in the calculation of two-loop amplitudes and subtraction terms
  - Subtraction method gives a mapping for phase space, which can be used for a more efficient numerical integration
- Human resources:
  - Team work, flat hierarchies
  - Project time scale $>$ post-doc contract

# Summary and Outlook

- Progress in the evaluation of two-loop amplitudes.

  Integration-by-parts, Lorentz identities and form-factor relations, Hopf algebra of nested sums.

- Progress in the cancellation of infrared divergences.

  Subtraction method at NNLO.

- Development in computer algebra.

  Object-oriented, open source.