NNLO corrections to three-jet production in electron-positron annihilation

Stefan Weinzierl

Universität Mainz

Introduction: **The strong coupling constant**

I.: **One- and two-loop amplitudes**

II: **Cancellation of divergences**

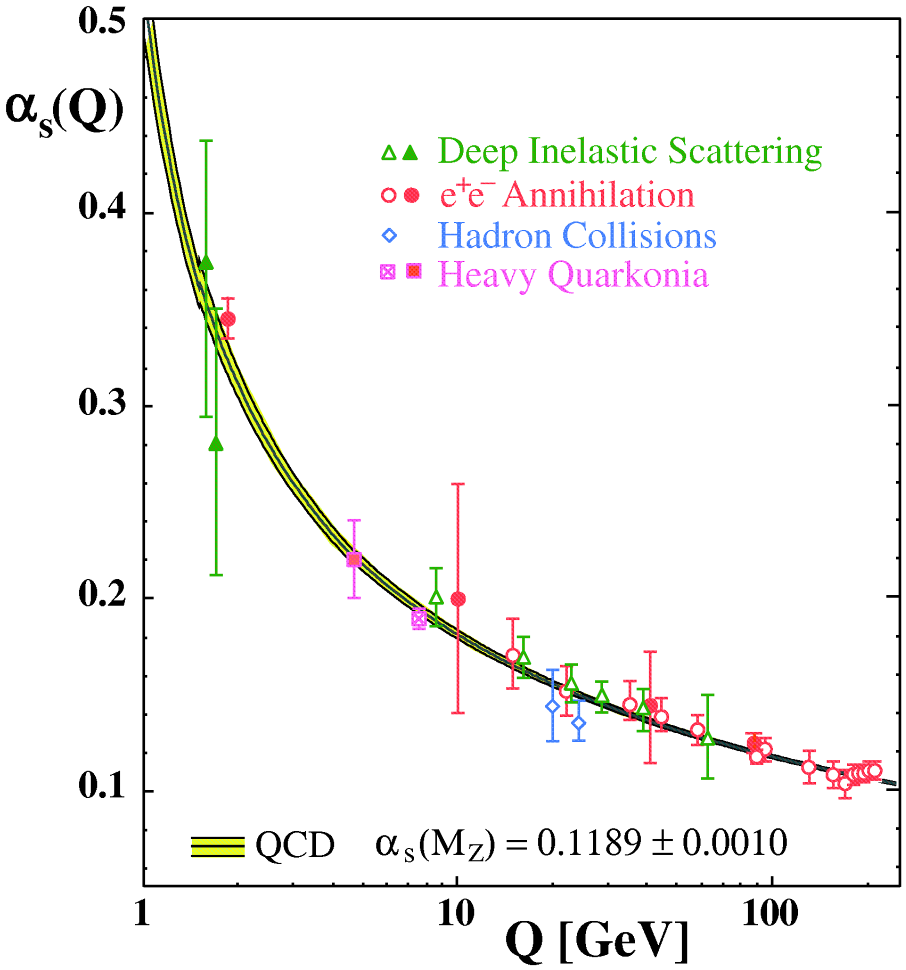
III: **Numerical results**

The strong coupling

The fundamental parameter of QCD:
The strong coupling α_s.

Objectives for LHC:
Extract fundamental quantities like α_s to high precision.

α_s can be measured in a variety of processes:
Deep inelastic scattering, τ-decays, heavy quarkonium, electron-positron annihilation, hadron collisions, ...



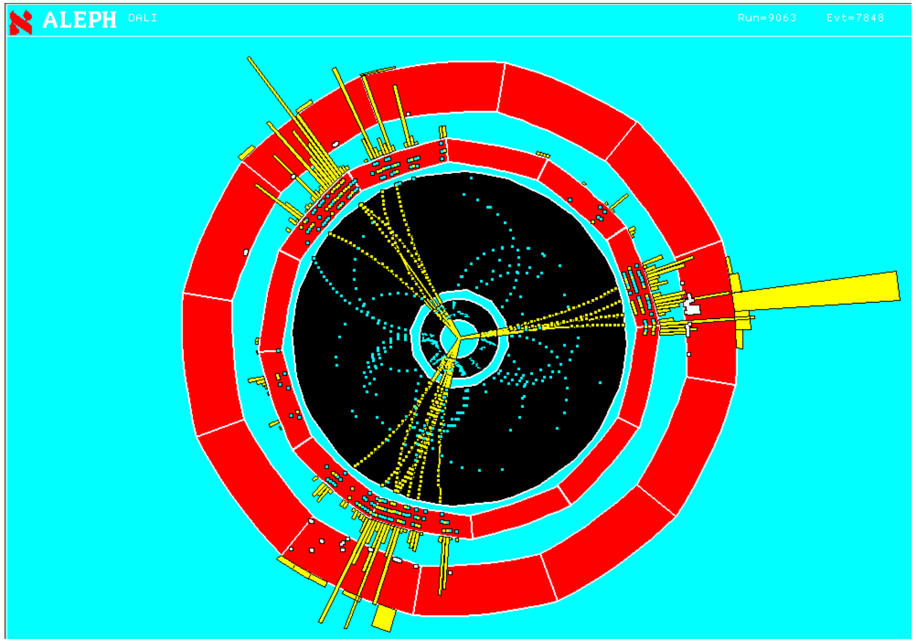
(S. Bethke, '06.)

The strong coupling from electron-positron annihilation

One possibility: Extract α_s from three-jet events in electron-positron annihilation.

Jets: A bunch of particles moving in the same direction

A three-jet event from the Aleph experiment at LEP:

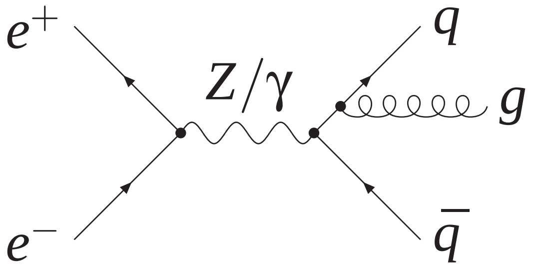


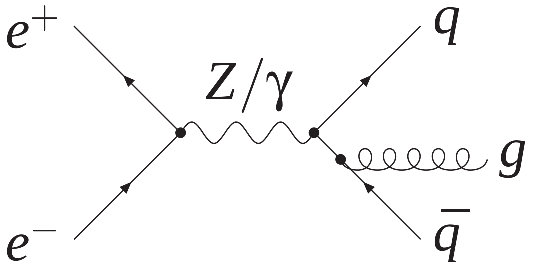
Perturbation theory

Due to the smallness of the coupling constants α and α_s, we may compute an observable at high energies reliable in perturbation theory,

$$\langle O \rangle = \frac{\alpha_s}{2\pi}\langle O \rangle_{LO} + \left(\frac{\alpha_s}{2\pi}\right)^2 \langle O \rangle_{NLO} + \left(\frac{\alpha_s}{2\pi}\right)^3 \langle O \rangle_{NNLO} + \dots$$

Feynman diagrams contributing to the leading order:



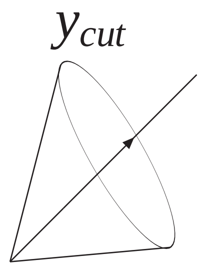


Leading order proportional to α_s !

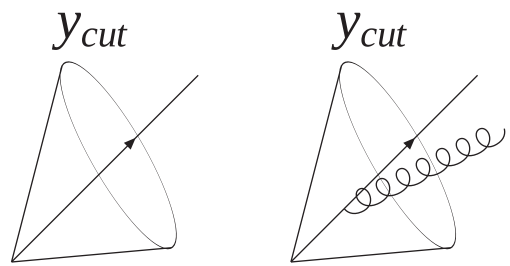
Modeling of jets:

In a perturbative calculation jets are modeled by only a few partons. This improves with the order to which the calculation is done.

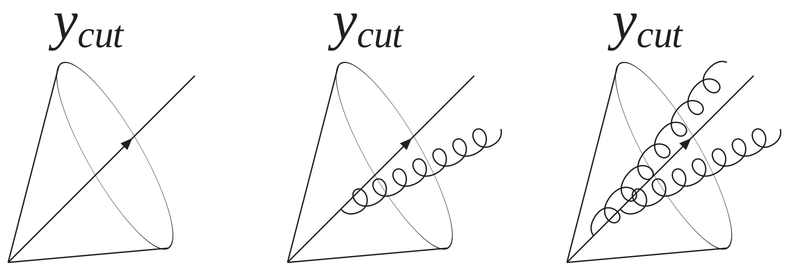
At leading order:



At next-to-leading order:



At next-to-next-to-leading order:



The master formula for the calculation of observables

$$\langle O \rangle = \underbrace{\frac{1}{2K(s)}}_{\text{flux factor}} \underbrace{\frac{1}{(2J_1+1)} \frac{1}{(2J_2+1)}}_{\text{average over initial spins}} \sum_n \underbrace{\int d\phi_{n-2}}_{\text{integral over phase space}} O(p_1, \ldots, p_n) \sum_{\text{helicity}} \underbrace{|\mathcal{A}_n|^2}_{\text{amplitude}}$$

Phase-space integration performed numerically by Monte-Carlo methods.

Observable infrared-safe: $O_{n+1}(p_1, \ldots, p_{n+1}) \longrightarrow O_n(p'_1, \ldots, p'_n),$ (Single unresolved)

$O_{n+2}(p_1, \ldots, p_{n+2}) \longrightarrow O_n(p'_1, \ldots, p'_n).$ (Double unresolved)

Amplitudes $\mathcal{A}_n$ calculated in perturbation theory.

Calculation of observables

Perturbative expansion of the amplitude (LO, NLO, NNLO):

$$|\mathcal{A}_n|^2 = \underbrace{\mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(0)}}_{\text{Born}} + \underbrace{\left(\mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(1)} + \mathcal{A}_n^{(1)\,*}\mathcal{A}_n^{(0)}\right)}_{\text{virtual}} + \underbrace{\left(\mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(2)} + \mathcal{A}_n^{(2)\,*}\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(1)\,*}\mathcal{A}_n^{(1)}\right)}_{\text{two-loop and loop-loop}},$$

$$|\mathcal{A}_{n+1}|^2 = \underbrace{\mathcal{A}_{n+1}^{(0)\,*}\mathcal{A}_{n+1}^{(0)}}_{\text{real}} + \underbrace{\left(\mathcal{A}_{n+1}^{(0)\,*}\mathcal{A}_{n+1}^{(1)} + \mathcal{A}_{n+1}^{(1)\,*}\mathcal{A}_{n+1}^{(0)}\right)}_{\text{loop+unresolved}},$$

$$|\mathcal{A}_{n+2}|^2 = \underbrace{\mathcal{A}_{n+2}^{(0)\,*}\mathcal{A}_{n+2}^{(0)}}_{\text{double unresolved}}.$$

$\mathcal{A}_n^{(l)}$: amplitude with n external particles and l loops.

Challenges

What are the bottle-necks ?

- Length: Perturbative calculations lead to expressions with a huge number of terms.
- Integrals: At one-loop and beyond, the occuring integrals cannot be simply looked up in an integral table.
- Divergences: At NLO and beyond, infrared divergences occur in intermediate stages, if massless particles are involved.
- Numerics: Stable and efficient numerical methods are required for the Monte Carlo integration.

Part I : One- and two-loop amplitudes

- One-loop amplitudes
- Two-loop integrals
- Polylogarithms

The amplitudes for $e^+e^- \longrightarrow 3$ jets at NNLO

A NNLO calculation of $e^+e^- \longrightarrow 3$ jets requires the following amplitudes:

- Born amplitudes for $e^+e^- \longrightarrow 5$ jets:

  F. Berends, W. Giele and H. Kuijf, 1989;

  K. Hagiwara and D. Zeppenfeld, 1989.

- One-loop amplitudes for $e^+e^- \longrightarrow 4$ jets:

  Z. Bern, L. Dixon, D.A. Kosower and S.W., 1996;

  J. Campbell, N. Glover and D. Miller, 1996.

- Two-loop amplitudes for $e^+e^- \longrightarrow 3$ jets:

  L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis and E. Remiddi, 2002;

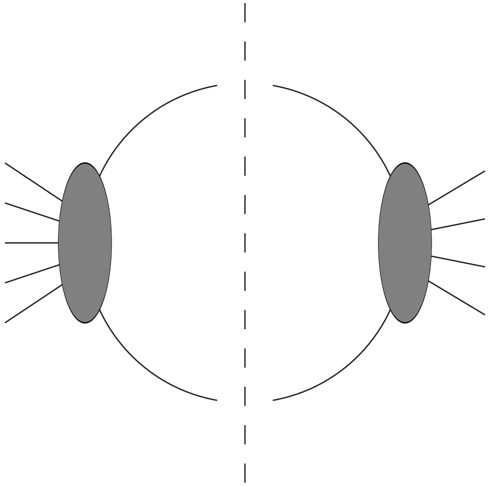
  S. Moch, P. Uwer and S.W., 2002.

Unitarity method

Problem: The one-loop amplitudes for $e^+e^- \longrightarrow 4$ partons depend on many kinematical invariants s_{ij}, resulting in lengthy expressions.

The cut-construction simplifies the calculation of one-loop amplitudes, as cancellations occur already inside A_L^{tree} and A_R^{tree}.

Bern, Dixon, Dunbar and Kosower, 1994



$$A^{1-loop} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2 + i\varepsilon} \frac{1}{k_2^2 + i\varepsilon} A_L^{tree} A_R^{tree}$$
$$+ \text{cut free pieces}$$

The cut technique has recently been refined: Prospects for multi-leg NLO calculation for the LHC.

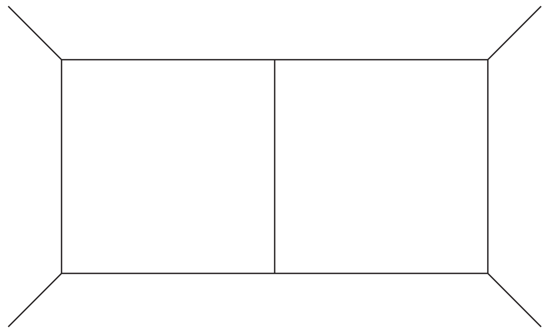
Britto, Cachazo, Feng, Bern, Dixon, Kosower, Forde, Berger, Mastrolia, Anastasiou, Kunszt, Ossola, Papadopoulos, Pittau, Bidder, Bjerrum-Bohr, Dunbar, ...

The calculation of two-loop integrals

- Techniques to calculate two-loop integrals
  - Mellin-Barnes transformation, Smirnov ’99, Tausk ’99.
  - Differential equations, Gehrmann, Remiddi ‘00.
  - Nested sums, Moch, Uwer, S.W. ‘01.
  - Sector decomposition (numerical), Binoth, Heinrich, ’00.
- Methods to reduce the work-load:
  - Integration-by-parts, Chetyrkin, Kataev, Tkachov ‘81.
  - Reduction algorithms, Tarasov ‘96, Laporta ’01.
  - Cut technique Bern, Dixon, Kosower, ’00

The double-box integral

Two-loop amplitudes for $2 \longrightarrow 2$ processes involve the double-box integral:



- First calculated by Smirnov (planar) and Tausk (non-planar) in 1999.
- Calculation based on Mellin-Barnes representation.
- Result expressed in harmonic polylogarithms.

$$H_{m_1,\ldots,m_k}(x) \quad = \quad \sum_{i_1>i_2>\ldots>i_k>0} \frac{x^{i_1}}{i_1^{m_1} i_2^{m_2} \ldots i_k^{m_k}}, \qquad x = \frac{s}{t}.$$

Multiple polylogarithms

- Definition:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

(Goncharov; Borwein, Bradley, Broadhurst and Lisonek)

- Special subsets: Harmonic polylogs, Nielsen polylogs, classical polylogs
(Remiddi and Vermaseren, Gehrmann and Remiddi).
- Have also an integral representation.
- Obey two Hopf algebras (Moch, Uwer, S.W.).
- Can be evaluated numerically for all complex values of the arguments
(Gehrmann and Remiddi, Vollinga and S.W.).

The integral representation for multiple polylogarithms

Define the functions G by

$$G(z_1,\ldots,z_k;y) \quad = \quad \int\limits_0^y \frac{dt_1}{t_1 - z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2 - z_2} \ldots \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Scaling relation:

$$G(z_1,\ldots,z_k;y) \quad = \quad G(xz_1,\ldots,xz_k;xy)$$

Short hand notation:

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) \quad = \quad G(\underbrace{0,\ldots,0}_{m_1-1},z_1,\ldots,z_{k-1},\underbrace{0\ldots,0}_{m_k-1},z_k;y)$$

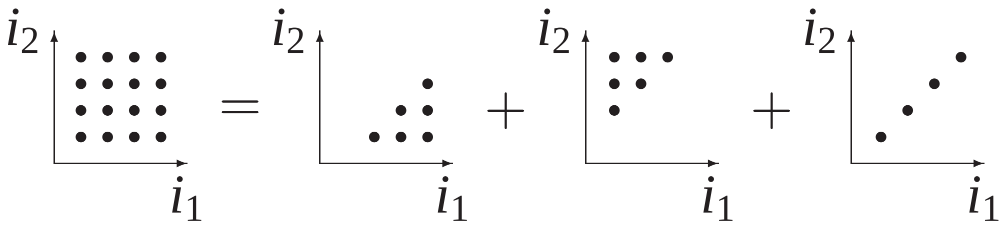
Conversion to the previous noation:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) \quad = \quad (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1 x_2},\ldots,\frac{1}{x_1\ldots x_k};1\right).$$

Shuffle algebra versus quasi-shuffle algebra

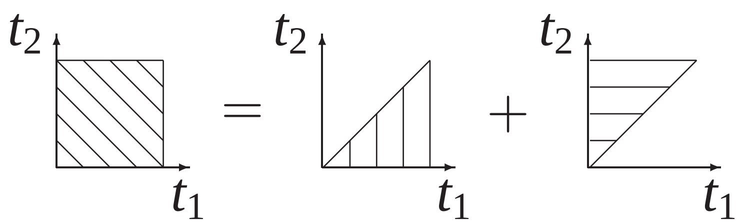
Quasi-shuffle algebra from the sum representation:

$$\text{Li}_{m_1}(x_1)\text{Li}_{m_2}(x_2) = \text{Li}_{m_1,m_2}(x_1,x_2) + \text{Li}_{m_2,m_1}(x_2,x_1) + \text{Li}_{m_1+m_2}(x_1x_2).$$



Shuffle algebra from the integral representation:

$$G(z_1;y)G(z_2;y) = G(z_1,z_2;y) + G(z_2,z_1;y)$$



The calculation of two-loop amplitudes

- Calculation of two-loop amplitudes
  - Bhabha, Bern, Dixon, Ghinculov '01.
  - $pp \longrightarrow$ 2 jets, Anastasiou, Glover, Oleari, Tejeda-Yeomans '01; Bern, De Freitas, Dixon, Ghinculov, Wong '01.
  - $e^+e^- \longrightarrow$ 3 jets, L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis and E. Remiddi '02; S. Moch, P. Uwer and S.W. '02
  - Higgs production, Harlander, Kilgore; Catani, de Florian, Grazzini; Anastasiou, Melnikov;
  - Drell-Yan, Anastasiou, Dixon, Melnikov, Petriello; Ravindran, Smith, van Neerven
- Calculation of three-loop splitting functions S. Moch, J. Vermaseren and A. Vogt '04;

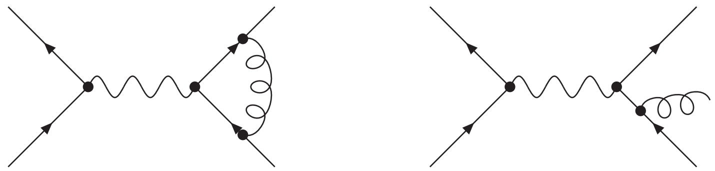
Part II : Cancellation of divergences

- Infrared divergences at NLO
- Infrared divergences at NNLO
- Soft gluons

Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, loop integrals can have infrared divergences.

For each IR divergence there is a corresponding divergence with the opposite sign in the real emission amplitude, when particles becomes soft or collinear (e.g. unresolved).



The Kinoshita-Lee-Nauenberg theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

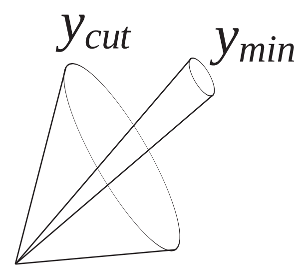
- Phase space slicing
  - e^+e^-: W. Giele and N. Glover, (1992)
  - initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
  - massive partons, fragmentation: S. Keller and E. Laenen, (1999)

- Subtraction method
  - residue approach: S. Frixione, Z. Kunzst and A. Signer, (1995)
  - dipole formalism: S. Catani and M. Seymour, (1996)
  - massive partons: L. Phaf and S.W. (2001), S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

Phase space slicing

Splits the integration of the real emission contribution into a region $y > y_{min}$ and a region $y < y_{min}$.

The former is free of singularities and the integration can be performed numerically there.
In the latter the matrix element is approximated and the integration over the one-parton phase space is performed analytically.



- Introduces an error of order y_{min}.
- The first region gives a contribution of the form

$$a \ln^2 y_{min} + b \ln y_{min} + c$$

  The logarithms $\ln^2 y_{min}$ and $\ln y_{min}$ cancel against the contribution from the second region.
- But: Cancelation happens only numerically!

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V = \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself.
  Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\epsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

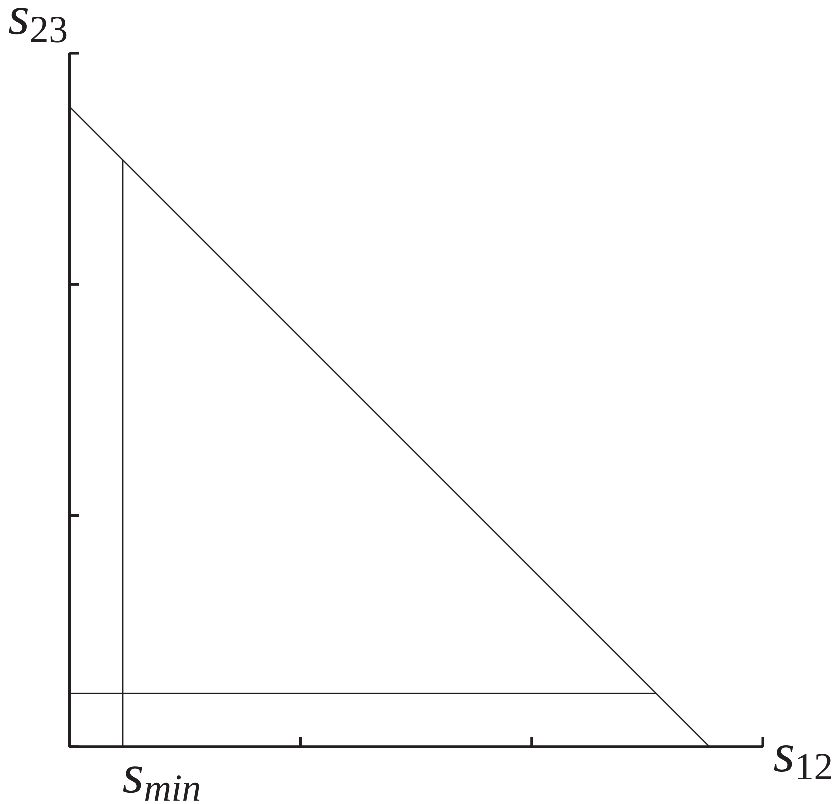
An example: $e^+e^- \longrightarrow$ 2 jets at NLO

The matrix element squared for $\gamma^* \rightarrow qg\bar{q}$:

$$M_3 = 8(1-\varepsilon)\left[2\frac{s_{123}^2}{s_{12}s_{23}} - 2\frac{s_{123}}{s_{12}} - 2\frac{s_{123}}{s_{23}} + (1-\varepsilon)\frac{s_{23}}{s_{12}} + (1-\varepsilon)\frac{s_{12}}{s_{23}} - 2\varepsilon\right]$$

The dipole subtraction terms:

$$\mathcal{D}_{12,3} + \mathcal{D}_{32,1} = 8(1-\varepsilon)$$
$$\left\{\left[2\frac{s_{123}^2}{s_{12}(s_{12}+s_{23})} - 2\frac{s_{123}}{s_{12}} + (1-\varepsilon)\frac{s_{23}}{s_{12}}\right]\right.$$
$$\left. + \left[2\frac{s_{123}^2}{s_{23}(s_{12}+s_{23})} - 2\frac{s_{123}}{s_{23}} + (1-\varepsilon)\frac{s_{12}}{s_{23}}\right]\right\}$$



The antenna subtraction term:

$$\mathcal{A}_{123} = \mathcal{D}_{12,3} + \mathcal{D}_{32,1}$$

Spin and colour correlations

- In the soft limit, amplitudes factorize completely in spin space, but colour correlations remain.
- In the collinear limit, amplitudes factorize completely in colour space, but spin correlations remain.

Spin-correlations occur for the splittings $g \longrightarrow gg$ and $g \longrightarrow q\bar{q}$, but not for $q \longrightarrow qg$.

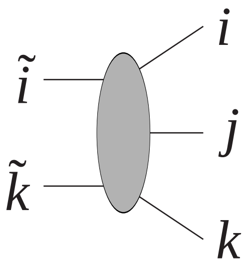
If one uses spin-averaged subtraction terms, one has a local counterterm only after the average over the azimuthal angle.

Alternative: Use combination of subtraction and slicing.

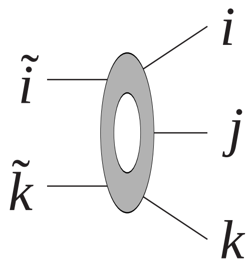
The subtraction method at NNLO

- Singular behaviour
  - Factorization of tree amplitudes in double unresolved limits, Berends, Giele, Cambell, Glover, Catani, Grazzini, Del Duca, Frizzo, Maltoni, Kosower ’99
  - Factorization of one-loop amplitudes in single unresolved limits, Bern, Del Duca, Kilgore, Schmidt, Kosower, Uwer, Catani, Grazzini, ’99
- Extension of the subtraction method to NNLO Kosower; S.W.; Kilgore; Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Frixione, Grazzini; Somogyi, Trócsányi and Del Duca;
- Cancellation based on sector decomposition Anastasiou, Melnikov, Petriello; Heinrich;
- Applications:
  - $pp \longrightarrow W$, Anastasiou, Dixon, Melnikov, Petriello ’03,
  - $pp \longrightarrow H$, Anastasiou, Dixon, Melnikov, Petriello ’05, Catani, Grazzini ’08
  - $e^+e^- \longrightarrow$ 2 jets, Anastasiou, Melnikov, Petriello ’04, S.W. ’06
  - $e^+e^- \longrightarrow$ 3 jets, Gehrmann-De Ridder, Gehrmann, Glover, Heinrich ’07, S.W. ’08

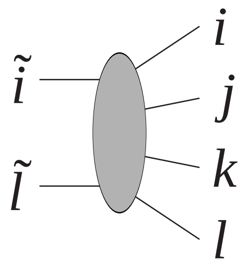
Antenna subtraction terms



NLO



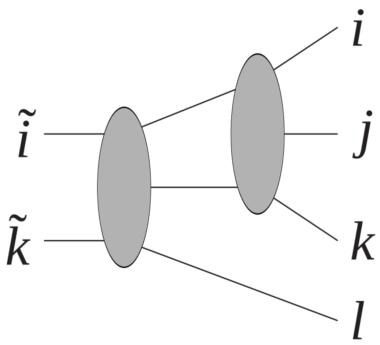
one-loop unresolved



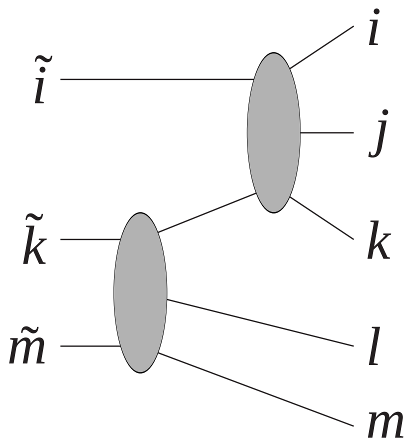
double unresolved

Gehrmann-De Ridder, Gehrmann, Glover, ’05

At NNLO also iterated structures:



colour connected



almost colour connected

The subtraction method at NNLO

$$\text{5 partons :} \quad d\sigma_5^{(0)} - d\alpha^{NLO} - d\alpha^{NNLO} + d\alpha^{iterated} - d\alpha^{almost} - d\alpha^{soft},$$

$$\text{4 partons :} \quad d\sigma_4^{(1)} + d\alpha^{NLO} - d\alpha^{loop} - d\alpha^{iterated} - d\alpha^{product} + d\alpha^{almost} + d\alpha^{soft},$$

$$\text{3 partons :} \quad d\sigma_3^{(2)} + d\alpha^{NNLO} + d\alpha^{loop} + d\alpha^{product}.$$

$d\alpha^{NNLO}$ contains the four-parton antenna functions,

$d\alpha^{almost}$ contains a product of two three-parton antenna functions,

$d\alpha^{iterated}$ is the approximation of $d\alpha^{NLO}$,

$d\alpha^{loop}$ is the approximation of the one-loop matrix elements,

$d\alpha^{product}$ contains a product of two three-parton antenna functions, both with $4 \rightarrow 3$ parton kinematics

$d\alpha^{soft}$ is an additional subtraction term due to soft gluons, occuring in processes with three or more hard partons.

Soft gluons

4 partons:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \ln\left(\frac{(1+c_j)(1-c_2)}{2(1-c_2c_j-s_2s_j\cos\phi)}\right) =$$

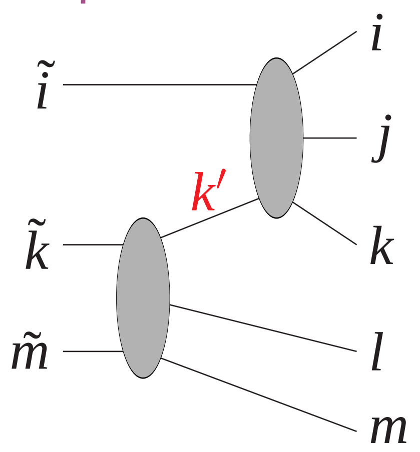
$$= \ln\left(\frac{1-c_2c_j+(c_j-c_2)}{1-c_2c_j+|c_j-c_2|}\right).$$

Non-zero for $c_j < c_2$!

The explicit poles in the four-parton configuration have to cancel: $d\alpha^{soft}$ is needed.

The five-parton contribution has to be independent of the slicing parameter: $-d\alpha^{soft}$ is needed.

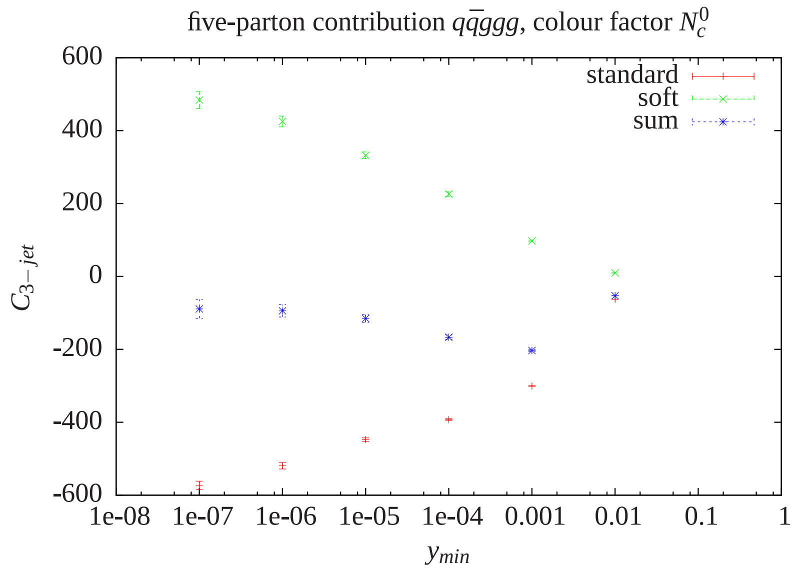
5 partons:



Gluon l soft:

Eikonal factor

$\mathrm{Eik}(k', l, m)$



Part III : Numerical results

- Infrared-safe observables
- Eventshapes and jets
- Results

Infrared-safe observables and event shapes

Observables which do not depend on long-distance behaviour, are called infrared-safe observables and can reliably be calculated in perturbation theory.

Example: Thrust

$$T = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_i |\vec{p}_i|}$$

In particular, it is required that they do not change value, if infinitesimal soft or collinear particles are added.

At NNLO:

$$\begin{aligned} \text{Single unresolved}: \quad & O_{n+1}(p_1, ..., p_{n+1}) & \rightarrow & \quad O_n(p_1', ..., p_n'), \\ \text{Double unresolved}: \quad & O_{n+2}(p_1, ..., p_{n+2}) & \rightarrow & \quad O_n(p_1', ..., p_n'). \end{aligned}$$

Jet algorithms

Ingredients:

- a resolution variable y_{ij} where a smaller y_{ij} means that particles i and j are "closer";

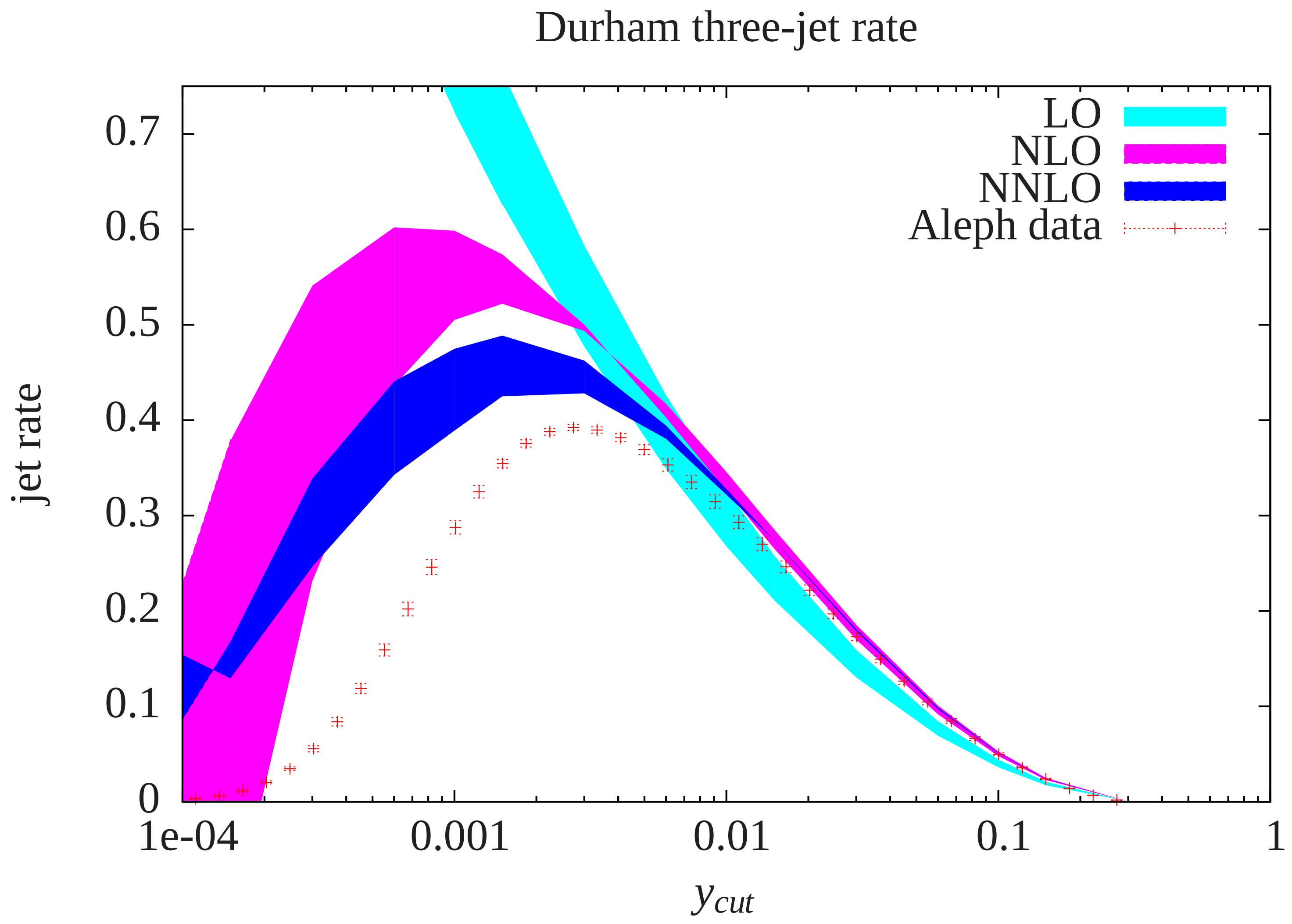
$$y_{ij}^{DURHAM} = \frac{2(1-\cos\theta_{ij})}{Q^2}\,\mathrm{min}(E_i^2, E_j^2)$$

- a combination procedure which combines two four-momenta into one;

$$p_{(ij)}^{\mu} = p_i^{\mu} + p_j^{\mu}.$$

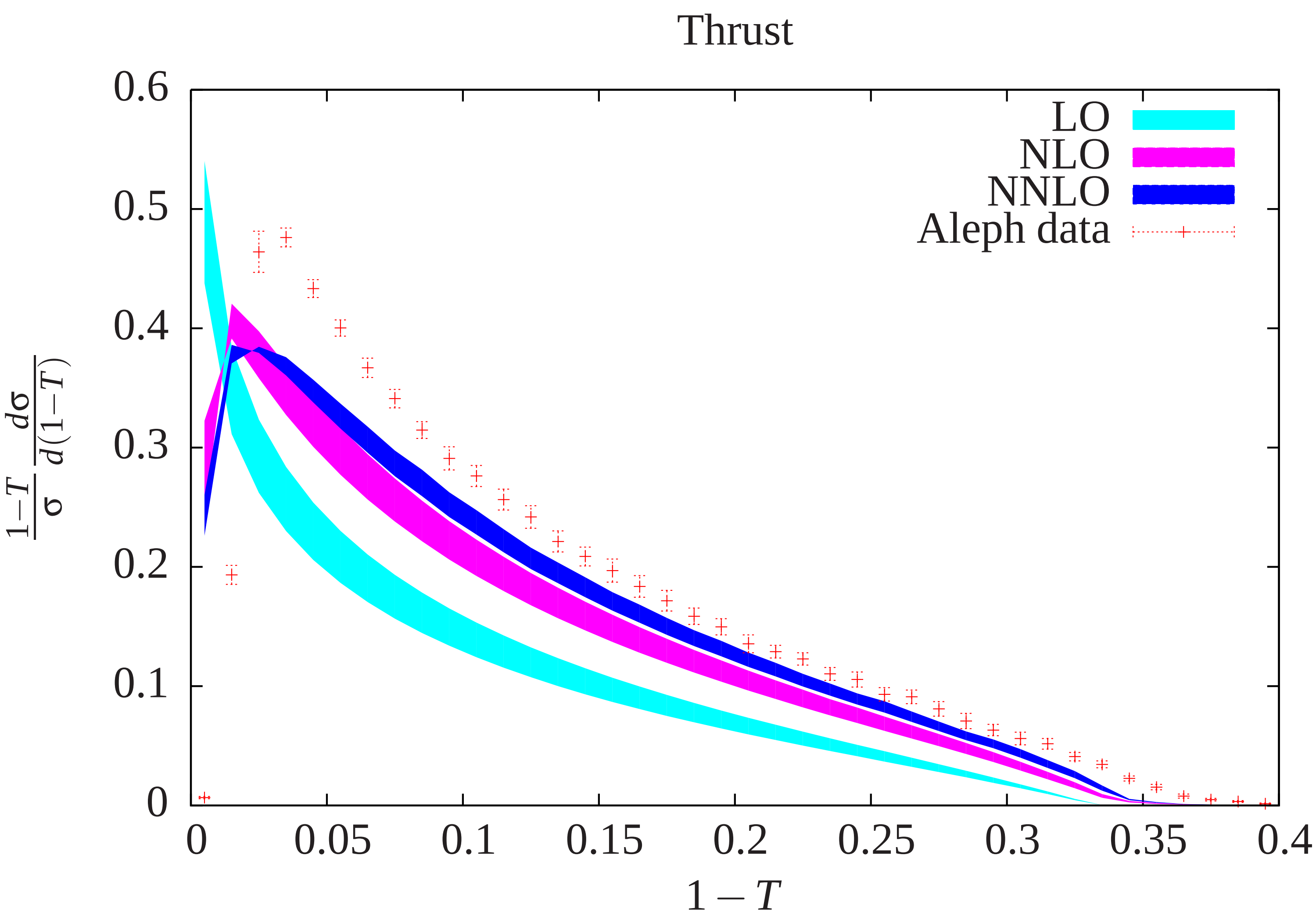
- a cut-off y_{cut} which provides a stopping point for the algorithm.

Results for the three-jet rate in electron-positron annihilation



S.W., arXiv:0807.3241

Results for the thrust distribution



preliminary !

Summary

- $e^+e^- \longrightarrow$ 3 jets at NNLO very useful for α_s
- Second independent calculation
- Challenging aspects due to three hard coloured particles