Matrix elements and loops
(Automated NLO calculations)

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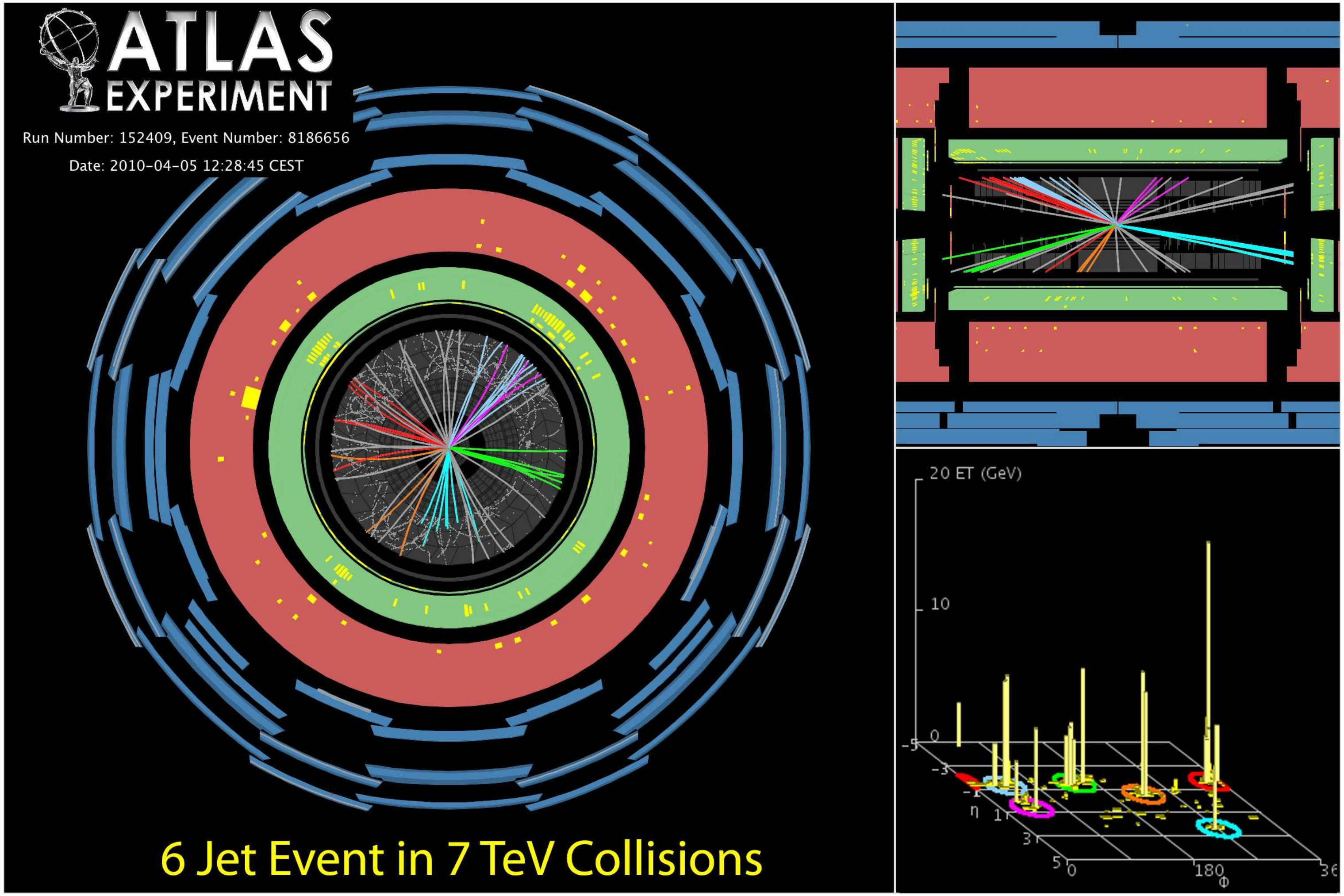
I: **Why textbook methods don't work**

II: **Common techniques**

III: **Differences between various approaches**

LHC physics



Jets: A bunch of particles moving in the same direction

Multileg NLO calculations

What one aims for: Accuracy and precision

- NLO calculations for multi-parton processes at the LHC.
  Multi-parton processes: 3, 4, 5, 6, ... partons in the final state.

- For a given process the program should be usable for any infrared-safe observable.

- Need to compute the Born, the virtual corrections and the real corrections.

The master formula for the calculation of observables

$$\langle O \rangle = \sum_{a,b} \underbrace{\int dx_1\, f_a(x_1) \int dx_2\, f_b(x_2)}_{\text{pdf's}} \underbrace{\frac{1}{2K(\hat{s})}}_{\text{flux factor}} \underbrace{\frac{1}{n_a^{\text{spin}} n_b^{\text{spin}} n_a^{\text{colour}} n_b^{\text{colour}}}}_{\text{average over initial spins and colours}}$$

$$\times \sum_n \underbrace{\int d\phi_n}_{\text{integral over phase space}} \underbrace{O(p_1, \ldots, p_n)}_{\text{observable}} \underbrace{|\mathcal{A}_{n+2}|^2}_{\text{amplitude}}$$

Phase-space integration performed numerically by Monte-Carlo methods.

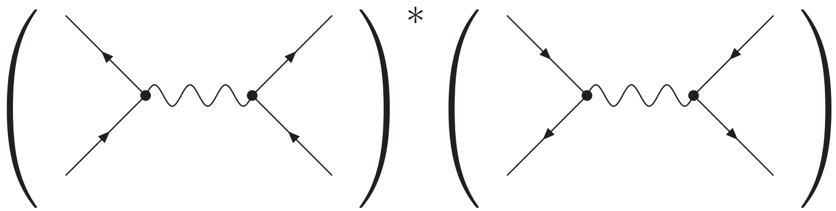
Observable infrared-safe: $O_{n+1}(p_1, \ldots, p_{n+1}) \to O_n(p'_1, \ldots, p'_n)$ for unresolved limit

Amplitudes $\mathcal{A}_n$ calculated in perturbation theory.

Perturbation theory

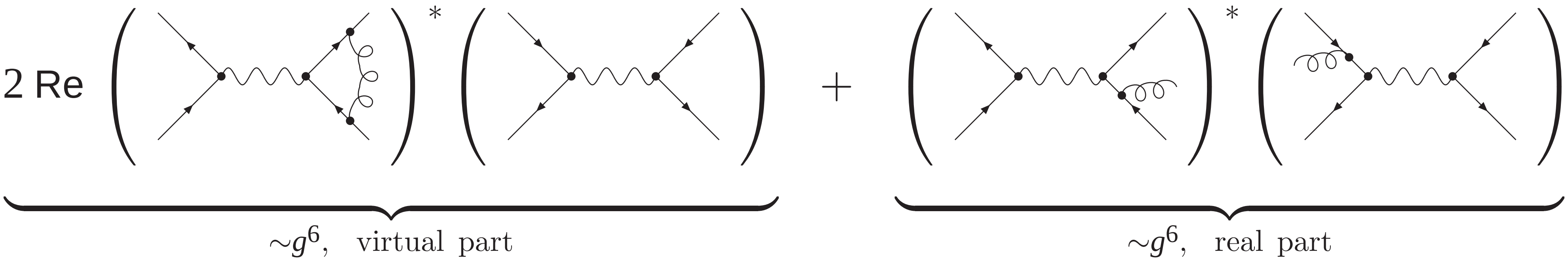
We need the amplitude squared:

At leading order (LO) only Born amplitudes contribute:



$\sim \quad g^4$

At next-to-leading order (NLO): One-loop amplitudes and Born amplitudes with an additional parton.



Real part contributes whenever the additional parton is not resolved.

Inconveniences we know to handle

- Loop amplitudes may have ultraviolet and infrared (soft and collinear) divergences.
- Dimensional regularisation is the method of choice for the regularisation of loop integrals.
- Ultraviolet divergences are removed by renormalisation.
- Phase space integration for the real emission diverges in the soft or collinear region.
- Unitarity requires the same regularisation (i.e. dimensional regularisation) for these divergences.
- Infrared divergences cancel between real and virtual contribution, or with an additional collinear counterterm in the case of initial-state partons.

The textbook method

- The amplitude is given as a sum of Feynman diagrams.
- Squaring the amplitude implies summing over spins and colour.
- One-loop tensor integrals can always be reduced to scalar integrals (Passarino-Veltman).
- All scalar integrals are known.
- Phase space slicing or subtraction method to handle infrared divergences.

Works in principle, but not in practice ...

An analogy: Testing prime numbers

To check if an integer N is prime,

- For $2 \leq j \leq \sqrt{N}$ check if j divides N.
- If such a j is found, N is not prime.
- Otherwise N is prime.

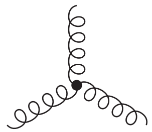
Works in principle, but not in practice ...

Brute force

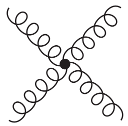
Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

2	4
3	25
4	220
5	2485
6	34300
7	559405
8	10525900

Feynman rules:



$$= g f^{abc} \left[(k_2 - k_3)_\mu g_{\nu\lambda} + (k_3 - k_1)_\nu g_{\lambda\mu} + (k_1 - k_2)_\lambda g_{\mu\nu} \right]$$



$$= -ig^2 \left[f^{abe} f^{ecd} \left(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right) + f^{ace} f^{ebd} \left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\rho} g_{\lambda\nu} \right) + f^{ade} f^{ebc} \left(g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} \right) \right]$$

Feynman diagrams are not the method of choice !

Helicity amplitudes

Suppose that an amplitude is given as the sum of N Feynman diagrams.
To calculate the amplitude squared à la Bjorken-Drell:
Sum over all spins and use

$$\begin{aligned}
\sum_\lambda \varepsilon^*_\mu(k,\lambda)\varepsilon_\nu(k,\lambda) &= -g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{kn}, \\
\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) &= \not{p} + m, \\
\sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) &= \not{p} - m.
\end{aligned}$$

This gives of the order N^2 terms.

Better: For each spin configuration evaluate the amplitude to a complex number. Taking the norm of a complex number is a cheap operation.

Spinors

Spinors are solutions of the Dirac equation.

For massless particles two-component Weyl spinors are a convenient choice:

$$|p+\rangle = \frac{1}{\sqrt{|p_+|}} \begin{pmatrix} -p_{\perp^*} \\ p_+ \end{pmatrix} \qquad |p-\rangle = \frac{1}{\sqrt{|p_+|}} \begin{pmatrix} p_+ \\ p_\perp \end{pmatrix}$$

$$\langle p+| = \frac{1}{\sqrt{|p_+|}} (-p_\perp, p_+) \qquad \langle p-| = \frac{1}{\sqrt{|p_+|}} (p_+, p_{\perp^*})$$

Light-cone coordinates: $p_+ = p_0 + p_3$, $p_- = p_0 - p_3$, $p_\perp = p_1 + ip_2$, $p_{\perp^*} = p_1 - ip_2$

Spinor products:

$$\langle pq \rangle = \langle p - | q+ \rangle\,, \qquad [qp] = \langle q + | p- \rangle\,.$$

The spinor products are anti-symmetric.

The spinor helicity method

Gluon polarisation vectors:

$$\varepsilon_\mu^+(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

q is an arbitrary light-like reference momentum. Dependency on q drops out in gauge invariant quantities.

Berends, Kleiss, De Causmaecker, Gastmans and Wu; Xu, Zhang and Chang;

Kleiss and Stirling; Gunion and Kunszt

Integration over helicity angles

Example: For $gg \to 7g$ we have $N = 559405$ Born diagrams.

- Helicity amplitudes reduce the complexity from

$$N^2 = 312933954025 \text{ terms}$$
$$\text{to } 2^n \cdot N = 512 \cdot 559405 \text{ terms.}$$

- Factor $2^n = 2^9 = 512$ from sum over all helicities.

- Replace sum over helicities by Monte Carlo integration over helicity angles:

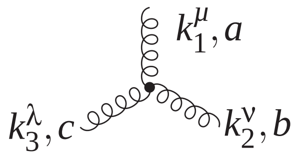
  P. Draggiotis, R. Kleiss, C. Papadopoulos, '98

$$\sum_{\lambda=\pm} \varepsilon_\mu^{\lambda *} \varepsilon_\nu^\lambda \quad = \quad \frac{1}{2\pi} \int\limits_0^{2\pi} d\phi \, \varepsilon_\mu(\phi)^* \varepsilon_\nu(\phi), \qquad \varepsilon_\mu(\phi) = e^{i\phi} \varepsilon_\mu^+ + e^{-i\phi} \varepsilon_\mu^- .$$

- Monte Carlo error is independent of the number of dimensions, this removes the factor 2^n.

Colour decomposition

Each Feynman rule has a colour part and a kinematical part:



$$= g \underbrace{f^{abc}}_{\text{colour}} \underbrace{\left[g^{\mu\nu} \left(k_1^\lambda - k_2^\lambda \right) + g^{\nu\lambda} \left(k_2^\mu - k_3^\mu \right) + g^{\lambda\mu} \left(k_3^\nu - k_1^\nu \right) \right]}_{\text{kinematic}}$$

In an amplitude collect all terms with the same colour structure.

Example: The n-gluon amplitude:

$$\mathcal{A}_n^{(0)}(g_1, g_2, \dots, g_n) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \underbrace{2 \, \mathrm{Tr} \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}} \right)}_{\text{colour factors}} \underbrace{A_n^{(0)} \left(g_{\sigma(1)}, \dots, g_{\sigma(n)} \right)}_{\text{partial amplitudes}}.$$

The partial amplitudes do not contain any colour information and are gauge-invariant. Each partial amplitude has a fixed cyclic order of the external legs.

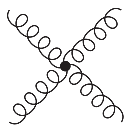
P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach; F. A. Berends and W. Giele; M. L. Mangano, S. J. Parke, and Z. Xu; D. Kosower, B.-H. Lee, and V. P. Nair; Z. Bern and D. A. Kosower.

Improvement due to the colour decomposition

Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

n	2	3	4	5	6	7	8
unordered	4	25	220	2485	34300	559405	10525900
cyclic ordered	3	10	36	133	501	1991	7335

Feynman rules: Four-gluon vertex



Traditional (unordered):

$$-ig^2 \left[f^{abe} f^{ecd} \left(g^{\mu\lambda} g^{\nu\rho} - g^{\mu\rho} g^{\nu\lambda} \right) + f^{ace} f^{ebd} \left(g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\lambda\nu} \right) + f^{ade} f^{ebc} \left(g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho} \right) \right]$$

Colour-stripped and cyclic ordered:

$$i \left(2 g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\nu\lambda} \right)$$

Symmetric phase space integration

Example: $q\bar{q} \to ng$ with colour decomposition:

$$\mathcal{A}_{n+2}^{(0)}(q,g_1,\ldots,g_n,\bar{q}) = g^n \sum_{\sigma\in S_n} \left(T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}}\right)_{i_q j_{\bar{q}}} A_n^{(0)}\left(q,g_{\sigma(1)},\ldots,g_{\sigma(n)},\bar{q}\right) = g^n \sum_{i=1}^{n!} C_i\, A_{n,i}^{(0)}.$$

There are $n!$ partial amplitudes.

Leading colour contribution:

$$\left|\mathcal{A}_{n+2}^{(0)}\right|^2_{\mathrm{lc}} = g^{2n} \sum_{i=1}^{n!} \left(C_i^\dagger C_i\right) \left|A_{n,i}^{(0)}\right|^2.$$

Phase space integration is symmetric, can remove sum with $n!$ terms:

$$\int d\phi_n O_n \left|\mathcal{A}_{n+2}^{(0)}\right|^2_{\mathrm{lc}} = n!\, g^{2n} \left(C_1^\dagger C_1\right) \int d\phi_n O_n \left|A_{n,1}^{(0)}\right|^2.$$

Colour decomposition at one-loop

One-loop amplitudes (and Born amplitudes with multiple quark pairs):

Partial amplitudes can be decomposed further into primitive amplitudes.

Z. Bern, L. Dixon, D. Kosower, '95

Primitive amplitudes defined by:

- fixed cyclic ordering of the QCD partons
- definite routing of the external fermion lines through the diagram
- particle content circulating in the loop

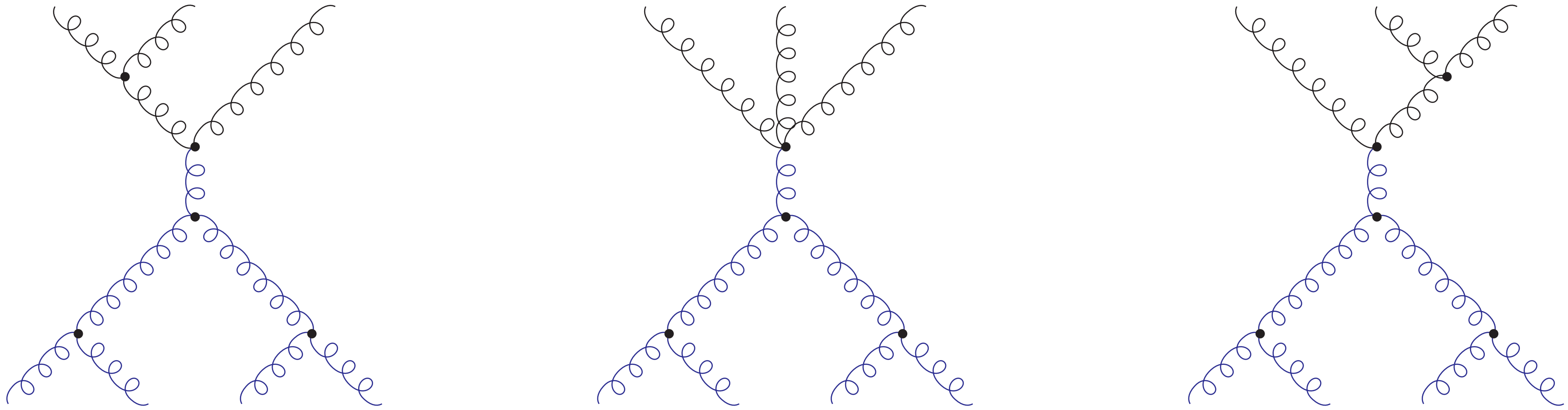
Systematic (and non-trivial) procedure to decompose partial amplitudes into primitive amplitudes:

Ellis, Giele, Kunszt, Melnikov, Zanderighi, '09,

Ita, Ozeren, '11,

Badger Biedermann, Uwer, Yundin, '12

How to avoid to compute the same sub-expression again and again

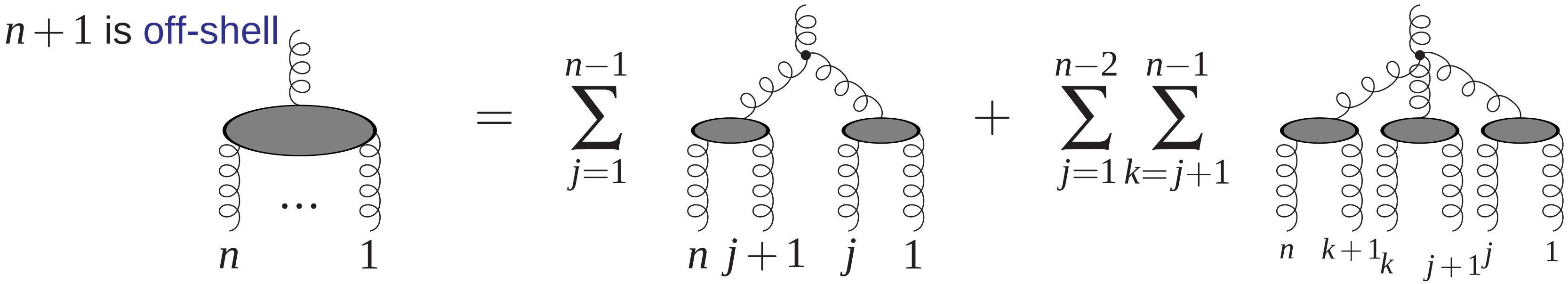


Lower part identical in all three diagrams.

Strategy: Compute this sub-expression once and store the result.

Recurrence relations

Off-shell currents $J^{\mu}(g_1,...,g_n)$ provide an efficient way to calculate amplitudes:



Momentum conservation: $p_{n+1} = p_1 + p_2 + ... + p_n$.

On-shell condition for particles 1 to n: $p_j^2 = m_j^2$.

Recursion start: $J^{\mu}(g_1) = \varepsilon_1^{\mu}$.

No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele,

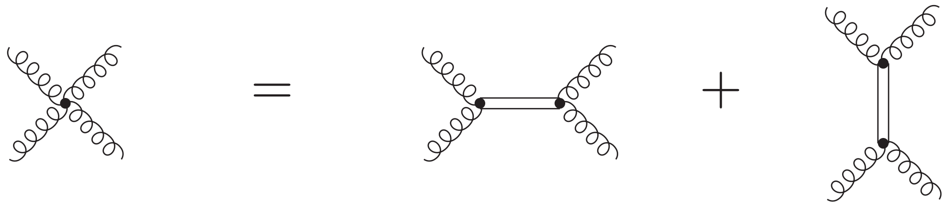
D. A. Kosower.

Computational costs

Born amplitudes with n particles and three- and four-valent vertices scale as n^4.

Can replace four-gluon vertex by a tensor particle, obtain only three-valent vertices:

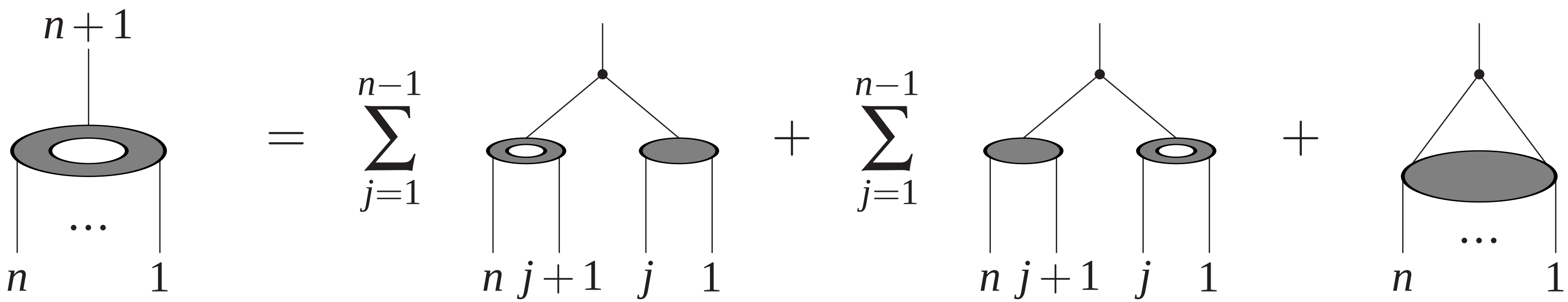
C. Duhr, S. Höche, F. Maltoni, ’06



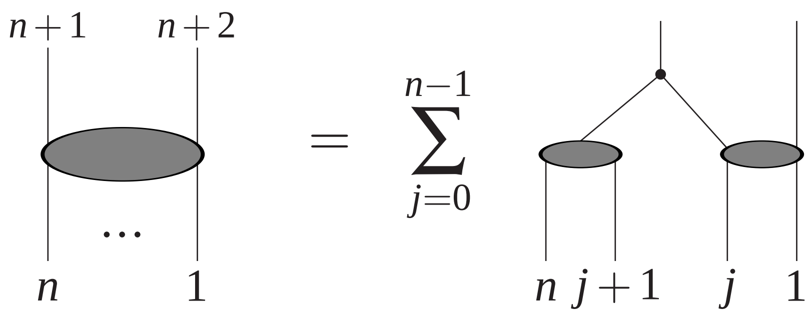
Scaling reduced to n^3.

Recurrence relations at one-loop

With only three-valent vertices we have for the integrand of a one-loop amplitude:

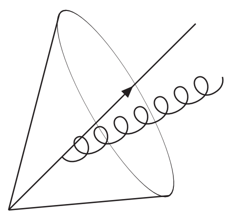


Recurrence relation for new tree-like object with two legs off-shell:



The real correction

- Born matrix element $|\mathcal{A}_{n+1}^{(0)}|^2$ with $(n+1)$ partons.
- Contributes whenever the additional parton is below y_{cut} and is not resolved.
- In particular this is the case in the soft and collinear region.
- Phase space integration over soft and collinear region diverges.



The Kinoshita-Lee-Nauenberg theorem

- The phase space integration over the unresolved region diverges, need a regulator.
- Unitarity requires the same regulator as in the virtual part, therefore use dimensional regularisation.
- Nature ensures that the amplitudes have a nice behaviour in the soft and collinear limits,
  physicists have to ensure that also the observables have a nice behaviour in these limits:
  Restriction to infrared-safe observables.
- For infrared-safe observables infrared divergences cancel in the sum of real and virtual corrections.
  This is the Kinoshita-Lee-Nauenberg theorem: Any infrared-safe observable, summed over all states degenerate according to some resolution criteria, will be finite.

The cancellation of infrared divergences in practise

- The real contribution has $(n+1)$ particles in the final state. In four space-time dimensions, the phase space integral is a $3(n+1)-4=3n-1$ dimensional integral.

- In $D=4-2\varepsilon$ space-time dimensions, the phase space integral is a

$$(D-1)(n+1)-D=3n-1-2n\varepsilon$$

dimensional integral.

- We want to perform the phase space integration by Monte Carlo techniques in four space-time dimensions.

The subtraction method

The NLO cross section is rewritten as

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V$$

$$= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

Variants of the subtraction method

The singular part of the subtraction terms is fixed, the finite part can be chosen freely.

- Residue subtraction: Frixione, Kunszt and Signer, ’95; Del Duca, Somogyi, Trócsányi, ’05; Frixione, ’11
- Dipole subtraction: Catani and Seymour ’96; Phaf and S.W. ’01; Catani, Dittmaier, Seymour and Trócsányi ’02; Dittmaier and Kasprzik, ’08; Czakon, Papadopoulos and Worek, ’09; Götz, Schwan, S.W., ’12
- Antenna subtraction: Kosower, ’97; Gehrmann-De Ridder, Gehrmann, Glover, ’05; Daleo, Gehrmann, Maitre, ’06; Gehrmann-De Ridder, Ritzmann, ’09
- Nagy-Soper subtraction (modified dipole subtraction) Nagy and Soper, ’07; Chung, Kramer and Robens, ’10; Bevilacqua, Czakon, Kubocz and Worek, ’13

Real emission (minus the subtraction terms) can be automated.

S.W., ’05, T. Gleisberg and F. Krauss, ’07, M. Seymour and C. Tevlin, ’08, K. Hasegawa, S. Moch and P. Uwer, ’08, R. Frederix, T. Gehrmann and N. Greiner, ’08, M. Czakon, C. Papadopoulos and M. Worek, ’09.

The virtual correction

- Tensor reduction technique:

  - At one-loop can always reduce tensor integrals to scalar integrals
  - Avoid Gram determinants
  - Recursive techniques can be used through open loops

- Cut-based techniques:

  - Scalar integrals are known, need only the coefficients of these integrals
  - Coefficients can be obtained by calculating tree-like objects
  - Have to solve a linear system of equations numerically
  - Need also rational terms not accompagnied by a scalar integral

- Numerical integration with subtraction and contour deformation:

  - Integrand is simple close to singular regions
  - Fast, scales like a Born calculation
  - Monte Carlo error depends on the chosen contour

Reduction of tensor integrals

The Passarino-Veltman algorithm:

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{k_\mu k_\nu}{(k^2 - m_1^2)((k-p_1)^2 - m_2^2)((k-p_1-p_2)^2 - m_3^2)}$$

$$= \quad p_1^\mu p_1^\nu C_{21} + p_2^\mu p_2^\nu C_{22} + (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) C_{23} + g^{\mu\nu} C_{24}.$$

Inverting the linear system of equations introduces Gram determinants:

$$\Delta \quad = \quad \left| \begin{array}{cc} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{array} \right| .$$

Improved algorithms avoid these Gram determinants!

A. Denner and S. Dittmaier,

T. Binoth, J.-Ph. Guillet, G. Heinrich, E. Pilon, C. Schubert,

F. del Aguila and R. Pittau,

A. van Hameren, J. Vollinga and S.W.,

F. Cascioli, P Maierhöfer, S. Pozzorini

Reduction of scalar integrals

Finite one-loop integrals with more than four propagators can always be reduced to integrals with maximally four propagators.

Melrose (1965)

Basic idea: In a space of dimension four there can be no more than four linear independet vectors.

The proof can be extended towards integrals computed within dimensional regularization.

Reduction of scalar integrals

Reduction of pentagons (W. van Neerven and J. Vermaseren; Z. Bern, L. Dixon, and D. Kosower):

$$I_5 = \sum_{i=1}^{5} b_i I_4^{(i)} + \mathcal{O}(\varepsilon) \, .$$

Reduction of hexagons (T. Binoth, J. P. Guillet, and G. Heinrich):

$$I_6 = \sum_{i=1}^{6} b_i I_5^{(i)} .$$

Reduction of scalar integrals with more than six propagators (G. Duplancic and B. Nizic):

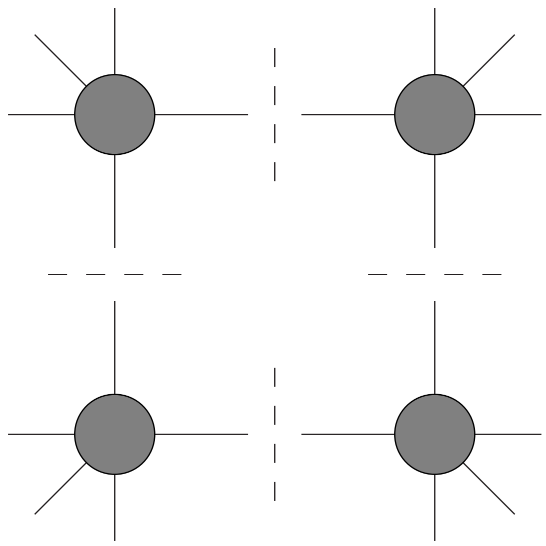
$$I_n = \sum_{i=1}^{n} r_i I_{n-1}^{(i)} .$$

Here, the decomposition is no longer unique.

Cut techniques

Scalar integrals are known, need only the coefficients in front and the rational part R_n:

$$A_n^{(1)} = \sum_{i,j,k,l} c_{ijkl} I_{ijkl}^{\text{Box}} + \sum_{i,j,k} c_{ijk} I_{ijk}^{\text{Triangle}} + \sum_{i,j} c_{ij} I_{ij}^{\text{Bubble}} + R_n$$



- Box coefficients from quadruple cuts.
- Triangle coefficients from triple cuts, after box contribution has been subtracted out.
- Bubble coefficients from double cuts, after box and triangle have been subtracted out.
- Rational part from cuts in D dimensions.

R. Britto, F. Cachazo, B. Feng; D. Forde; G. Ossola, C. Papadopoulos, R. Pittau; Anastasiou, Britto, Feng, Kunszt, Mastrolia; Ellis, Giele, Kunszt, Melnikov; Badger, Sattler, Yundin; ...

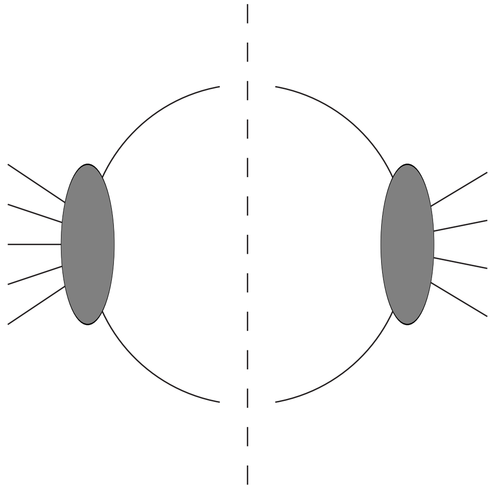
Cut techniques

Prehistoric version of the cut technique: Cutkosky rules

Cutkosky, '60

Medieval version of the cut technique:

$$A^{(1)} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2 + i\varepsilon} \frac{1}{k_2^2 + i\varepsilon} A_L^{(0)} A_R^{(0)} + \text{cut free pieces}$$



Bern, Dixon, Dunbar and Kosower, '94; Bern, Morgan, '95

First multi-particle one-loop amplitude calculated with this technique:

$$e^+ e^- \to 4 \text{ partons}$$

Bern, Dixon, Kosower, S.W., '96

Subtraction method for loop integrals

Use subtraction also for the virtual part:

$$\int_{n+1} d\sigma^R + \int_n d\sigma^V = \underbrace{\int_{n+1} \left(d\sigma^R - d\sigma^A\right)}_{\text{convergent}} + \underbrace{\int_n \left(\mathbf{I} + \mathbf{L}\right) \otimes d\sigma^B}_{\text{finite}} + \underbrace{\int_n \left(d\sigma^V - d\sigma^{A'}\right)}_{\text{convergent}}$$

- In the last term $d\sigma^V - d\sigma^{A'}$ the Monte Carlo integration is over a phase space integral of n final state particles plus a 4-dimensional loop integral.
- All explicit poles cancel in the combination $\mathbf{I} + \mathbf{L}$.
- Divergences of one-loop amplitudes related to IR-divergences (soft and collinear) and to UV-divergences.

M. Assadsolimani, S. Becker, D. Götz, Ch. Reuschle, Ch. Schwan, S.W.

Numerical NLO QCD calculations

Proceed through the following steps:

1. Local subtraction terms for soft, collinear and ultraviolet singular part of the integrand of one-loop amplitudes

2. Contour deformation for the 4-dimensional loop integral.

3. Numerical Monte Carlo integration over phase space and loop momentum.

Not a new idea: Nagy and Soper proposed in ’03 this method, working graph by graph.

(see also: Soper; Krämer, Soper; Catani et al.; Kilian, Kleinschmidt)

What is new: The IR-subtraction terms can be formulated at the level of amplitudes, no need to work graph by graph.

The IR-subtraction terms are universal and amasingly simple.

Recent results

Impressive list of results:

- $pp \to W + 5$ jets,
- $pp \to Z + 4$ jets,
- $pp \to WW + 2$ jets,
- $pp \to t\bar{t} + 2$ jets,
- $pp \to 4$ jets,
- $e^+e^- \to 7$ jets,

Berger et al. (Blackhat collaboration), Ellis, Melnikov, Zanderighi, Melia, Rontsch, Bevilacqua, Czakon, Pittau, Papadopoulos, Worek, Bredenstein, Denner, Dittmaier, Pozzorini, Frederix, Frixione, Badger, Biedermann, Uwer, Yundin, Becker, Götz, Reuschle, Schwan, S.W., ...

Computer programs

Many codes, some public, others not:

- Blackhat
- GoSam
- HELAC/CutTools
- Madloops/Madgraph
- NJet
- OpenLoops
- Recola
- Rocket
- Sherpa

Berger, Bern, Diana, Ozeren, Dixon, Höche, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre, Cullen, Greiner, Heinrich, Luisoni, Mastrolia, Mirabella, Ossola, Reiter, Tramontano, Bevilacqua, Czakon, Garzelli, van Hameren, Kardos, Malamos, Ossola, Papadopoulos, Pittau, Worek, Hirschi, Frederix, Frixione, Garzelli, Maltoni, Pittau, Torrielli, Badger, Biedermann, Uwer, Yundin, Cascioli, Maierhöfer, Pozzorini, Actis, Denner, Hofer, Scharf, Uccirati, Ellis, Melnikov, Zanderighi, Krauss, Schonherr, Siegert, ...

Summary and outlook

- Impressive revolution in our abilities to calculate NLO corrections.
- For the experimentalists: Talk to us and use the NLO calculations !