Relations and representations of QCD amplitudes

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I.: **Introduction**

II: **BCJ relations**, **CHY representation**, **KLT relations**

III: **Speculations**

in collaboration with L. de la Cruz and A. Kniss

L. de la Cruz, A. Kniss, S.W., "Proof of the fundamental BCJ relations for QCD amplitudes", JHEP 1509 (2015) 197,

L. de la Cruz, A. Kniss, S.W., "The CHY representation of tree-level primitive QCD amplitudes", JHEP 1511 (2015) 217,

L. de la Cruz, A. Kniss, S.W., "Double copies of fermions as only gravitational interacting matter", arXiv:1601.04523.

Introduction: QCD amplitudes

- Pure Yang-Mills theory ("gluons only"):
  - Self-interactions of massless gauge bosons.
- QCD ("gluons and quarks"):
  - Massless gauge bosons plus fermions in the fundamental representation of the gauge group.
  - Flavour is conserved.
  - The fermions may be massive.

Consider scattering amplitudes with n external particles (n_g gluons, n_q quarks, n_q anti-quarks).

$$n = n_g + 2n_q.$$

In this talk: No loops, only legs!

Colour decomposition of QCD amplitudes

QCD amplitudes may be decomposed into sums of group-theoretical factors multiplied by kinematic functions called primitive amplitudes.

Example: The n-gluon tree amplitude:

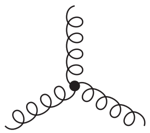
$$\mathcal{A}_n^{\mathrm{YM}}(1,2,...,n) \quad = \quad g^{n-2} \sum_{\sigma \in S_n / Z_n} \underbrace{2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)}_{\text{colour factors}} \; \underbrace{A_n^{\mathrm{YM}}\left(\sigma(1),...,\sigma(n)\right)}_{\text{primitive amplitudes}}.$$

Properties of the primitive amplitudes:

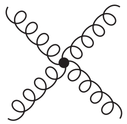
- All group-theoretical factors have been stripped off.
- The primitive amplitudes are gauge-invariant.
- Each primitive amplitude has a fixed cyclic order of the external legs.

Feynman rules

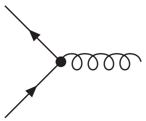
Primitive amplitudes are calculated from cyclic-ordered Feynman rules:



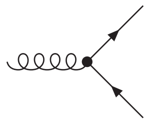
$$= i\left[g^{\mu_1\mu_2}\left(p_1^{\mu_3}-p_2^{\mu_3}\right)+g^{\mu_2\mu_3}\left(p_2^{\mu_1}-p_3^{\mu_1}\right)+g^{\mu_3\mu_1}\left(p_3^{\mu_2}-p_1^{\mu_2}\right)\right],$$



$$= i\left[2g^{\mu_1\mu_3}g^{\mu_2\mu_4}-g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right],$$



$$= i\gamma^\mu,$$



$$= -i\gamma^\mu.$$

Part I

BCJ relations

Pure Yang-Mills theory

How many independent primitive amplitudes are there for the n-gluon tree amplitude?

- There are $n!$ external orderings.
- Cyclic invariance reduce the number to $(n-1)!$.
- The Kleiss-Kuijf relations reduce the number to $(n-2)!$.

  Kleiss, Kuijf, 1989
- The Bern-Carrasco-Johansson relations reduce the number to $(n-3)!$.

  Bern, Carrasco, Johansson, 2008

Basis of independent amplitudes consists of $(n-3)!$ elements.

Relations among pure gluon amplitudes

- Cyclic invariance:

$$A_n^{\mathrm{YM}}(1,2,...,n) \quad = \quad A_n^{\mathrm{YM}}(2,...,n,1).$$

  Proof: trivial

- Kleiss-Kuijf relations:

$$A_n^{\mathrm{YM}}(1,\vec{\beta},2,\vec{\alpha}) \quad = \quad (-1)^{n-2-j} \sum_{\sigma\in\vec{\alpha}\,\sqcup\!\sqcup\,\vec{\beta}^T} A_n^{\mathrm{YM}}(1,2,\sigma_1,...,\sigma_{n-2}).$$

  Reason: anti-symmetry of the vertices

- Fundamental Bern-Carrasco-Johansson relations:

$$\sum_{i=2}^{n-1}\left(\sum_{j=i+1}^{n} 2p_2p_j\right) A_n^{\mathrm{YM}}(1,3,...,i,2,i+1,...,n-1,n) \quad = \quad 0.$$

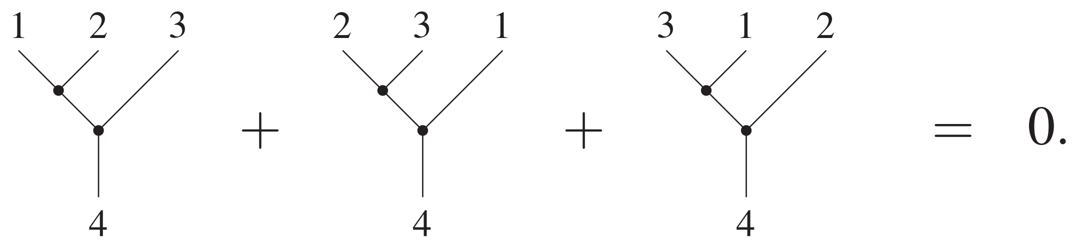
  Reason: Jacobi-like identities for kinematical numerators

Jacobi-like identities for kinematical numerators

Colour-kinematics duality states that gauge theory amplitudes can be brought in a form

$$\mathcal{A}_n^{\mathrm{YM}} = i g^{n-2} \sum_{\text{trivalent graphs } G} \frac{C(G)\,N(G)}{D(G)}, \qquad D(G) = \prod_{\text{edges } e} s_e,$$

where the kinematical numerators $N(G)$ satisfy Jacobi-like relations, whenever the corresponding colour factors $C(G)$ do:



$$C(G_1) + C(G_2) + C(G_3) = 0 \quad \Rightarrow \quad N(G_1) + N(G_2) + N(G_3) = 0$$

Bern, Carrasco, Johansson, 2010

Relations for primitive QCD amplitudes

Consider now primitive tree amplitudes with gluons and quarks. What are the relations?

- The trivial and obvious relations:
  - Cyclic invariance
  - Kleiss-Kuijf relations
  - No-crossed-fermion-lines relation (new!)

- The non-trivial relations:
  - Fundamental BCJ relations for gluons:

$$\sum_{i=2}^{n-1} \left(\sum_{j=i+1}^{n} 2 p_2 p_j \right) A_n^{\mathrm{QCD}}(1, 3, ..., i, 2_g, i+1, ..., n-1, n) \quad = \quad 0.$$

Relations for primitive QCD amplitudes

- BCJ relations for primitive QCD tree amplitudes:
  - Conjectured by H. Johansson and A. Ochirov, July 2015.
  - Proven by L. de la Cruz, A. Kniss and S.W., August 2015.
  - Proof based on BCFW recursion.

- Size of basis:

$$N_{\text{basis}} = \begin{cases} (n-3)!, & n_q \in \{0,1\}, \\ (n-3)!\frac{2\left(n_q-1\right)}{n_q!}, & n_q \geq 2. \end{cases}$$

- Example: 3 quark pairs:

$$A_6(q_1,q_2,q_3,\bar{q}_3,\bar{q}_2,\bar{q}_1),\; A_6(q_1,q_3,\bar{q}_3,q_2,\bar{q}_2,\bar{q}_1),\; A_6(q_1,q_3,q_2,\bar{q}_2,\bar{q}_3,\bar{q}_1),\; A_6(q_1,q_2,\bar{q}_2,q_3,\bar{q}_3,\bar{q}_1).$$

  Compare 6 gluons:

$$A_6(1,2,3,4,5,6),\; A_6(1,3,4,2,5,6),\; A_6(1,4,2,3,5,6),\; A_6(1,4,3,2,5,6),\; A_6(1,3,2,4,5,6),\; A_6(1,2,4,3,5,6).$$

Part II

CHY representation

The CHY representation of pure Yang-Mills amplitudes

The n-gluon tree amplitude with external ordering w and helicity configuration ε has a representation in the form of a global residue:

$$A_n^{\mathrm{YM}}(w,p,\varepsilon) \;=\; i \sum_{\text{solutions } j} J\left(z^{(j)},p\right) C\left(w,z^{(j)}\right) E\left(z^{(j)},p,\varepsilon\right).$$

The sum is over the inequivalent solutions $z=(z_1,z_2,\ldots,z_n)$ of the scattering equations

$$f_i(z,p) \;=\; \sum_{j=1,j\neq i}^{n} \frac{2p_i\cdot p_j}{z_i-z_j}.$$

The function $C\left(w,z^{(j)}\right)$ encodes the information on the external ordering,
the function $E\left(z^{(j)},p,\varepsilon\right)$ encodes the information on the external polarisations.

Cachazo, He, Yuan, 2013

The CHY representation of QCD amplitudes

Is there a similar representation for primitive QCD tree amplitudes, which separates the information on the ordering from the one on the polarisations? Yes!

$$A_n^{\mathrm{QCD}}(w,p,\varepsilon) = i \sum_{\text{solutions } j} J\left(z^{(j)},p\right) \hat{C}\left(w,z^{(j)}\right) \hat{E}\left(z^{(j)},p,\varepsilon\right).$$

- Construction of $\hat{C}\left(w,z^{(j)}\right)$ based on amplitude relations.
- Construction of $\hat{E}\left(z^{(j)},p,\varepsilon\right)$ based on pseudo-inverse matrices.
- Factorisation not unique.
- Valid for massless and massive quarks.

L. de la Cruz, A. Kniss and S.W., 2015

Part III

KLT relations

Perturbative gravity

Let us consider (small) fluctuations around the flat Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},$$

with $\kappa = \sqrt{32\pi G}$ and consider an effective theory defined by the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\mathrm{EH}} = -\frac{2}{\kappa^2}\sqrt{-g}R.$$

The field $h_{\mu\nu}$ describes a graviton.

The inverse metric $g^{\mu\nu}$ and $\sqrt{-g}$ are infinite series in $h_{\mu\nu}$, therefore

$$\mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{GF}} = \sum_{n=2}^{\infty} \mathcal{L}^{(n)},$$

where $\mathcal{L}^{(n)}$ contains exactly n fields $h_{\mu\nu}$.
Thus the Feynman rules will give an infinite tower of vertices.

Amplitudes for bi-adjoint scalars, gluons and gravitons

The n-particle double-ordered scalar amplitude m_n with three-valent vertices:

$$m_n(w, \tilde{w}, p) = i \sum_{\text{solutions } j} J\left(z^{(j)}, p\right) C\left(w, z^{(j)}\right) C\left(\tilde{w}, z^{(j)}\right)$$

The n-gluon primitive amplitude A_n^{YM}:

$$A_n^{\mathrm{YM}}(w, p, \varepsilon) = i \sum_{\text{solutions } j} J\left(z^{(j)}, p\right) C\left(w, z^{(j)}\right) E\left(z^{(j)}, p, \varepsilon\right)$$

The n-graviton amplitude M_n:

$$M_n(p, \varepsilon, \tilde{\varepsilon}) = i \sum_{\text{solutions } j} J\left(z^{(j)}, p\right) E\left(z^{(j)}, p, \varepsilon\right) E\left(z^{(j)}, p, \tilde{\varepsilon}\right)$$

Cachazo, He and Yuan, 2013

Graviton amplitudes from gluon amplitudes

- CHY representation:

$$M_n(p,\varepsilon,\tilde{\varepsilon}) = i \sum_{\text{solutions } j} J\left(z^{(j)},p\right) E\left(z^{(j)},p,\varepsilon\right) E\left(z^{(j)},p,\tilde{\varepsilon}\right)$$

Cachazo, He and Yuan, 2013

- Colour-kinematics duality:

$$M_n(p,\varepsilon,\tilde{\varepsilon}) = (-1)^{n-3} i \sum_{\text{trivalent graphs } G} \frac{N(G)N(G)}{D(G)}$$

Bern, Carrasco,Johansson, 2010

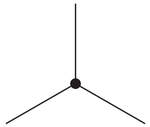
- KLT relations:

$$M_n(p,\varepsilon,\tilde{\varepsilon}) = -i \sum_{w,\tilde{w}\in B} A_n^{\text{YM}}(p,w,\varepsilon)\, S_{w\tilde{w}}\, A_n^{\text{YM}}(p,\tilde{w},\tilde{\varepsilon})$$

Kawai, Lewellen, Tye, 1986

KLT relations

Recall: $m_n(w, \tilde{w}, p)$ double-ordered scalar amplitude with three-valent vertices:



Define $(n-3)! \times (n-3)!$-dimensional matrix $m_{w\tilde{w}}$ by

$$m_{w\tilde{w}} = m_n(w, \tilde{w}, p).$$

Define KLT-matrix as the inverse of the matrix m:

$$S = m^{-1}$$

The matrix S enters

$$M_n(p, \varepsilon, \tilde{\varepsilon}) = -i \sum_{w, \tilde{w} \in B} A_n^{\mathrm{YM}}(p, w, \varepsilon) \, S_{w\tilde{w}} \, A_n^{\mathrm{YM}}(p, \tilde{w}, \tilde{\varepsilon})$$

Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove, 2010, Cachazo, He and Yuan, 2013, de la Cruz, Kniss, S.W., 2016

Generalisation

- Let's try the following:

$$
\begin{array}{ccc}
\text{gluon amplitudes} & \xrightarrow{\text{double copy}} & \text{graviton amplitudes} \\
\text{extension} \downarrow & & \\
\text{QCD amplitudes} & \xrightarrow{\text{double copy}} & ???
\end{array}
$$

- Double copy of QCD amplitudes:
  - Generalised KLT relations
  - Colour-kinematics duality

Generalised KLT relations

Let us now consider double-ordered amplitudes $m_n^{\mathrm{flav}}(w,\tilde{w},p)$ with un-flavoured massless scalars (as before) and flavoured scalars (massless or massive).

Flavour is conserved. Define $N_{\mathrm{basis}} \times N_{\mathrm{basis}}$-dimensional matrix $m_{w\tilde{w}}^{\mathrm{flav}}$ by

$$m_{w\tilde{w}}^{\mathrm{flav}} = m_n^{\mathrm{flav}}(w,\tilde{w},p).$$

Define generalised KLT-matrix as the inverse of the matrix m^{flav}:

$$S^{\mathrm{flav}} = \left(m^{\mathrm{flav}}\right)^{-1}$$

and

$$M_n^{\mathrm{method\ 1}}(p,\varepsilon,\tilde{\varepsilon}) = -i \sum_{w,\tilde{w}\in B} A_n^{\mathrm{QCD}}(p,w,\varepsilon)\, S_{w\tilde{w}}^{\mathrm{flav}}\, A_n^{\mathrm{QCD}}(p,\tilde{w},\tilde{\varepsilon})$$

de la Cruz, Kniss, S.W., 2016

Colour-kinematics duality for QCD amplitudes

Bring QCD amplitudes into the form

$$A_n^{\mathrm{QCD}}(p,w,\varepsilon) \;=\; i \sum_{G\in\mathcal{T}(w)} \frac{N(G)}{D(G)}, \qquad D(G) = \prod_{e\in E(G)} \left(s_e - m_e^2\right),$$

where the kinematical numerators $N(G)$ satisfy Jacobi-like relations, whenever the corresponding colour factors do.

Then

$$M_n^{\mathrm{method}\,2}(p,\varepsilon,\tilde{\varepsilon}) \;=\; (-1)^{n-3}\, i \sum_{G\in\mathcal{U}} \frac{N(G)\,N(G)}{D(G)}.$$

Johansson, Ochirov, 2014

Conjecture

- The two methods compute the same quantity:

$$M_n^{\mathrm{method}\,1}(p,\varepsilon,\tilde{\varepsilon}) \quad = \quad M_n^{\mathrm{method}\,2}(p,\varepsilon,\tilde{\varepsilon})\,.$$

- $M_n^{\mathrm{method}\,1/2}(p,\varepsilon,\tilde{\varepsilon})$ has properties of scattering amplitudes:
  - Invariant under (generalised) gauge transformations.
  - The only poles are single poles in the allowed factorisation channels.

- Evidence: All amplitudes with $n \leq 8$.

de la Cruz, Kniss, S.W., 2016

Spin states

Gluons and quarks have both two spin states, which we may label by $+$ and $-$.

Double copies of gluons and quarks have then four spin states:

$$++, \quad +-, \quad -+, \quad --\,.$$

Double copy of gluons:

Graviton corresponds to $++$ and $--$.

The states $+-$ and $-+$ correspond to a dilaton and an antisymmetric tensor.

Pure graviton amplitudes: No propagation of internal $+-$ or $-+$ states.

This is no longer true with massive flavours!

Part IV

Speculations

Interpretation

Let's re-insert the coupling:

$$\mathcal{M}_n(p,\varepsilon,\tilde{\varepsilon}) \quad = \quad \left(\frac{\kappa}{4}\right)^{n-2} M_n^{\mathrm{method}\ 1/2}(p,\varepsilon,\tilde{\varepsilon})\,, \qquad\qquad \kappa = \sqrt{32\pi G_N}.$$

- Particles interact with gravitational strength.
- Flavoured particles may be massive and non-relativistic.
- All Born scattering amplitudes may be computed.

This defines a model for massive non-relativistic particles interacting only with gravitational strength.

Comments

- Classical limit of massive amplitudes corresponds to an attractive $1/r$-potential. Effective coupling larger by a factor 2 due to exchange of $+-$ and $-+$ states.

- Might want to remove dilaton/antisymmetric tensor with the help of ghosts.

  Johansson, Ochirov, 2014

- All evindence on dark matter up to now gravitational.
  Most experimental searches assume additional weak-scale interactions.
  Model has only gravitational interactions.

- Open question: Explanation of the relic abundance.

- Independent of SUSY and strings.

Conclusions

- Basis of independent primitive QCD tree amplitudes: BCJ relations
- Separation helicity information / ordering information: CHY representation
- Double copies of QCD particles: KLT relations
- Double copies of massive fermions: “Dark matter amplitudes”
- ... many open questions ...