On-shell recursion relations

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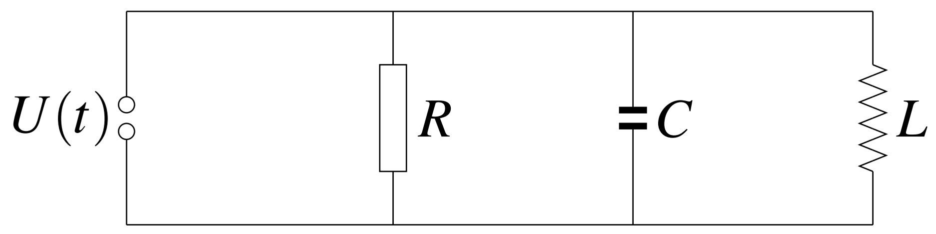
I.   **Techniques for many external legs**

II.   **Twistors, MHV vertices and recurrence relations**

III.   **Proof of the on-shell recurrence relations**

Prelude: Electrical circuits



Question: Given R, C, L and $U(t)$, what is the current $I(t)$ in the circuit ?

Remark 1: All quantities are real, the calculation can be performed within the real numbers, making extensive use of the addition theorems for $\sin$ and $\cos$.

Remark 2: The calculation simplifies if we view $U(t)$ and $I(t)$ as the real part of some complex functions.

Jet physics at the LHC

- Jet production: $pp \to$ jets
- Heavy flavour: $pp \to t\bar{t}+$ jets
  $pp \to t\bar{t}+W/Z/H+$ jets
- Single boson: $pp \to W/Z/\gamma+$ jets
- Diboson: $pp \to VV+$ jets

Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

2	4
3	25
4	220
5	2485
6	34300
7	559405
8	10525900

Feynman diagrams are not the method of choice !

Part I : Techniques for many external legs

- Colour decomposition
- Spinor methods
- Off-shell recurrence relations
- Parke-Taylor formulae

Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors carrying the colour structures multiplied by kinematic functions called partial amplitudes.

The partial amplitudes do not contain any colour information and are gauge-invariant. Each partial amplitude has a fixed cyclic order of the external legs.

Examples: The n-gluon amplitude:

$$\mathcal{A}_n(1,2,...,n) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \underbrace{2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)}_{\text{Chan Patton factors}} \underbrace{A_n\left(\sigma(1),...,\sigma(n)\right)}_{\text{partial amplitudes}}.$$

P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach,

F. A. Berends and W. Giele,

M. L. Mangano, S. J. Parke, and Z. Xu,

D. Kosower, B.-H. Lee, and V. P. Nair,

Z. Bern and D. A. Kosower.

The spinor helicity method

- Basic objects: Massless two-component Weyl spinors

$$|p\pm\rangle, \qquad \langle p\pm|$$

- Gluon polarization vectors:

$$\varepsilon_\mu^+(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

  q is an arbitrary light-like reference momentum. Dependency on q drops out in gauge invariant quantities.

- A clever choice of the reference momentum can reduce significantly the number of diagrams which need to be calculated.

Berends, Kleiss, De Causmaecker, Gastmans and Wu; Xu, Zhang and Chang;

Kleiss and Stirling; Gunion and Kunszt

Bra-ket notation versus dotted-undotted indices

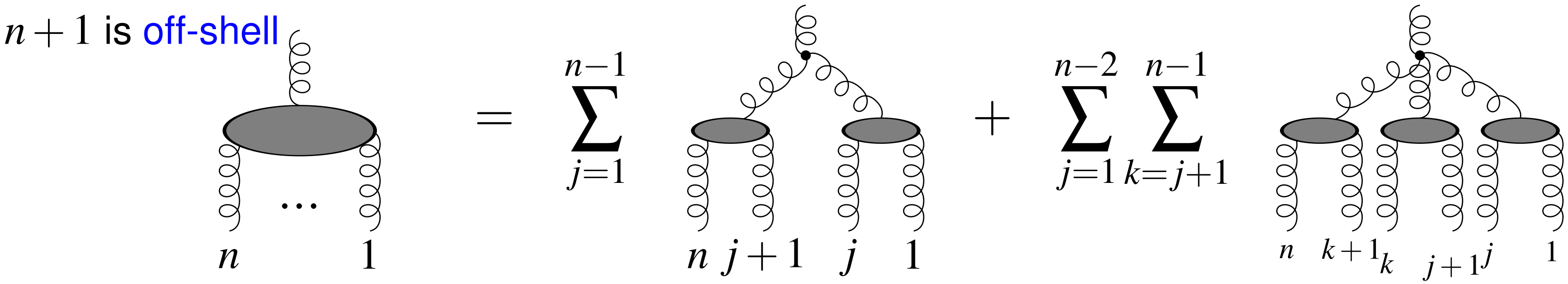
Two different notations for the same thing:

$$|p+\rangle = p_B \qquad \langle p+| = p_{\dot{A}}$$

$$|p-\rangle = p^{\dot{B}} \qquad \langle p-| = p^A$$

Recurrence relations

Off-shell currents provide an efficient way to calculate amplitudes:



Momentum conservation: $p_{n+1} = p_1 + p_2 + \ldots + p_n$.

On-shell condition for particles 1 to n: $p_j^2 = m_j^2$.

No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele,

D. A. Kosower.

The Parke-Taylor formulae

For specific helicity combinations the amplitudes have a remarkably simple analytic formula or vanish altogether:

$$\begin{aligned} A_n^{tree}(g_1^+, \ldots, g_n^+) &= 0, \\ A_n^{tree}(g_1^+, \ldots, g_j^-, \ldots, g_n^+) &= 0, \\ A_n^{tree}(g_1^+, \ldots, g_j^-, \ldots, g_k^-, \ldots, g_n^+) &= i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk \rangle^4}{\langle 12 \rangle \ldots \langle n1 \rangle}. \end{aligned}$$

The n-gluon amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity is called a maximal-helicity violating amplitude (MHV amplitude).

F. A. Berends and W. T. Giele,

S. J. Parke and T. R. Taylor.

Part II : Twistors, MHV vertices and recurrence relations

- Twistor space
- MHV vertices
- On-shell recursion relations

Twistor space

Each light-like vector has a bispinor representation:

$$p^\mu \quad \longrightarrow \quad p_A p_{\dot{B}}$$

Spinors only determined modulo the scaling

$$p_A \longrightarrow \lambda p_A, \qquad p_{\dot{B}} \longrightarrow \frac{1}{\lambda} p_{\dot{B}}.$$

Twistor space: Transform $p_{\dot{B}}$, but not p_A:

$$\begin{aligned} p_{\dot{A}} \quad &\longrightarrow \quad i\frac{\partial}{\partial q^{\dot{A}}}, \\ -i\frac{\partial}{\partial p^{\dot{A}}} \quad &\longrightarrow \quad q_{\dot{A}}. \end{aligned}$$

Twistor space continued

In signature $++--$, this transformation can be implemented as a Fourier transformation:

$$A\left(q^{\dot{A}}\right) = \int \frac{d^2p}{(2\pi)^2} \exp\left(iq^{\dot{A}} p_{\dot{A}}\right) A\left(p_{\dot{A}}\right).$$

In twistor space, the scaling relation reads

$$\left(p_A, q_{\dot{B}}\right) \rightarrow \left(\lambda p_A, \lambda q_{\dot{B}}\right).$$

Therefore twistor space is a three-dimensional projective space.

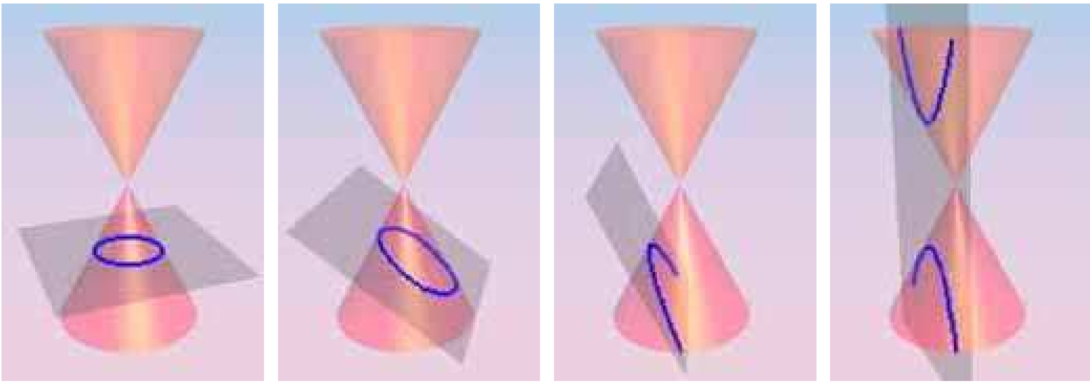
Algebraic curves

Examples of algebraic varieties: The cone is defined by

$$\{(x_1, x_2, x_3)\,; x_1^2 + x_2^2 - x_3^2 = 0\}\,.$$

A conic section is given by

$$\{(x_1, x_2, x_3)\,; x_1^2 + x_2^2 - x_3^2 = 0, ax_1 + bx_2 + cx_3 = 0\}\,.$$



Witten's conjecture

Witten conjectured that the n-gluon amplitude with l-loops is non-zero only if all points lie in twistor space on an algebraic curve of degree d. The degree d of this curve is given by the number of negative helicity gluons plus the number of loops minus one.

E. Witten, Commun. Math. Phys. 252, (2004), 189, (hep-th/0312171)

Imprecise statement: A curve of degree d is something like an instanton with topological charge d.
One instanton of charge d is equivalent to d instantons of charge 1.

The CSW construction

Cachazo, Svrček and Witten proposed that the gluonic Born amplitude with an arbitrary helicity configuration can be calculated from diagrams with scalar propagators and new vertices, which are MHV-amplitudes continued off-shell.

$$A_n(1^+, ..., j^-, ..., k^-, ..., n^+) \quad = \quad i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk \rangle^4}{\langle 12 \rangle ... \langle n1 \rangle}.$$

Off-shell continuation:

$$P \quad = \quad p^\flat + \frac{P^2}{2Pq} q.$$

Propagators are scalars:

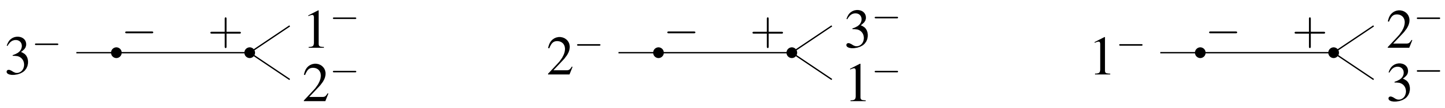
$$\frac{-i}{P^2}$$

Cachazo, Svrček and Witten, JHEP 0409:006, (hep-th/0403047)

Example: Six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

The first non-trivial example: The six-gluon amplitude with 3 positive helicity gluons and 3 negative helicity gluons.

One starts with stripped diagrams:



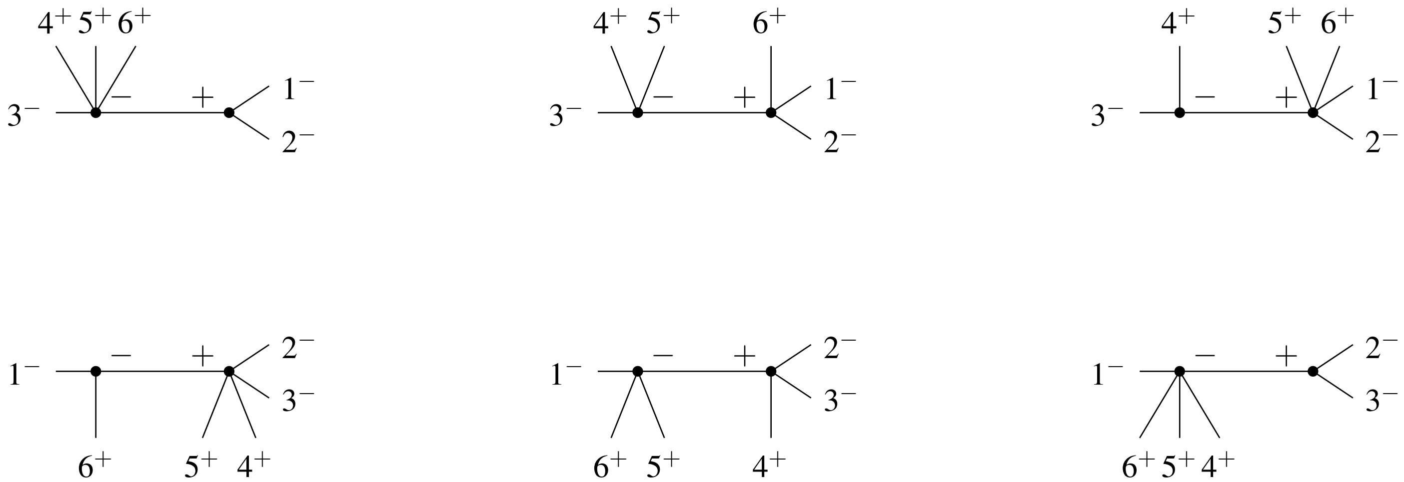
The second diagram will be dressed with all positive helicty gluons inserted between leg 3 and leg 1.

Therefore one MHV vertex with two negative helicity gluons and zero positive helicity gluons remains.

Therefore this diagram does not give a contribution.

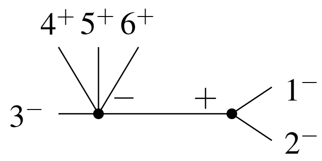
Example: Six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

Inserting the gluons with positive helicity:



Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

The first diagram yields:



$=$

$$\left[i\sqrt{2} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\left(-k_{12}^\flat\right)\rangle\langle\left(-k_{12}^\flat\right)1\rangle}\right] \frac{i}{k_{12}^2} \left[i\left(\sqrt{2}\right)^3 \frac{\langle 3k_{12}^\flat\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6k_{12}^\flat\rangle\langle k_{12}^\flat 3\rangle}\right]$$

Similar for the five other diagrams.

Compare this to

- a brute force approach (220 Feynman diagrams)

- colour-ordered amplitudes (36 diagrams)

On-shell recursion relations

Britto, Cachazo and Feng gave a recursion relation for the calculation of the n-gluon amplitude:

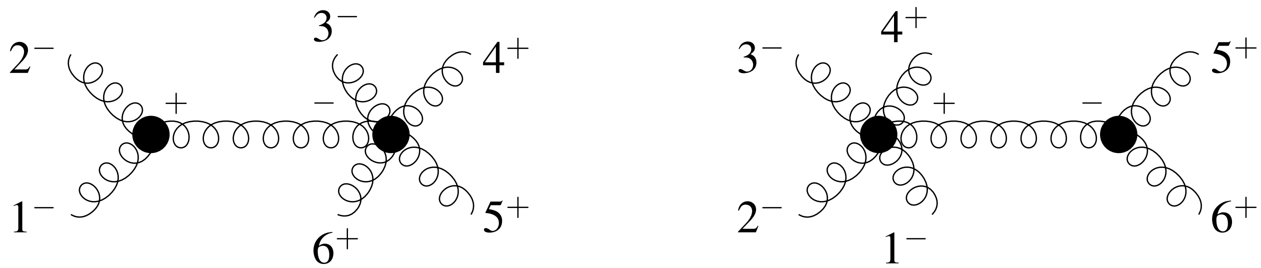
$$A_n\left(p_1, p_2, \dots, p_{n-1}^-, p_n^+\right) = \sum_{i=1}^{n-3} \sum_{\lambda=+,-} A_{i+2}\left(\hat{p}_n, p_1, p_2, \dots, p_i, -\hat{P}_{n,i}^{\lambda}\right)\left(\frac{i}{P_{n,i}^2}\right) A_{n-i}\left(\hat{P}_{n,i}^{-\lambda}, p_{i+1}, \dots, p_{n-2}, \hat{p}_{n-1}\right).$$

No off-shell continuation needed. The amplitudes on the r.h.s. are evaluated with shifted momenta.

Britto, Cachazo and Feng, Nucl. Phys. B715, (2005), 499, (hep-th/0412308)

Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

Only two diagrams contribute:



$$A_6^{tree}(1^+,2^+,3^+,4^-,5^-,6^-) =$$

$$4i\left[\frac{\langle 6-|1+2|3-\rangle^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{126}\langle 2-|1+6|5-\rangle} + \frac{\langle 4-|5+6|1-\rangle^3}{\langle 23\rangle\langle 34\rangle[56][61]s_{156}\langle 2-|1+6|5-\rangle}\right]$$

The number of diagrams

Example: Number of diagrams contributing to the colour-ordered six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$:

brute force approach:	220
colour-ordered amplitudes:	36
MHV vertices:	6
on-shell recursion:	2

Part III : Proof of the on-shell recurrence relations

- Momentum space versus spinor space
- Cauchy's residue theorem
- Vanishing at infinity

Britto, Cachazo, Feng and Witten, Phys. Rev. Lett. 94:181602, (2005), (hep-th/0501052),

Badger, Glover, Khoze and Svrcek, JHEP 07, (2005), 025, (hep-th/0504159)

Risager, JHEP 12, (2005), 003, (hep-th/0508206),

Draggiotis, Kleiss, Lazopoulos and Papadopoulos, Eur. Phys. J. C46, (2006), 74, (hep-ph/0511288),

Schwinn and S.W., JHEP 04, (2007) ,072, (hep-ph/0703021)

Momentum space versus spinor space

An amplitude is originally a function of a set of four-momenta $\{p_1, p_2, \ldots, p_n\}$.

Replace each four-vector p by two spinors p_A and $p_{\dot{A}}$.

Given the spinors one recovers the four-vector p^μ as follows:

$$p_\mu \quad = \quad \frac{1}{2} p_{\dot{A}} \bar{\sigma}_\mu^{\dot{A}B} p_B .$$

The amplitude is then a function of the spinors:

$$A_n\left(p_1^A, p_1^{\dot{B}}, \ldots, p_n^Y, p_n^{\dot{Z}}\right),$$

Shifting the spinors

Single out two particles for special treatment.

Shift p_i^A and $p_j^{\dot{A}}$, while p_j^A and $p_i^{\dot{A}}$ remain unchanged:

$$p_i^{A\prime} = p_i^A - z p_j^A, \qquad p_i^{\dot{A}\prime} = p_i^{\dot{A}},$$
$$p_j^{A\prime} = p_j^A, \qquad p_j^{\dot{A}\prime} = p_j^{\dot{A}} + z p_i^{\dot{A}}.$$

The shifted spinors correspond to on-shell particles with four-momenta

$$p_i^{\prime\mu} = p_i^\mu - \frac{z}{2} p_{i\,\dot{A}} \bar{\sigma}^{\mu\,\dot{A}B} p_{j\,B}, \qquad p_j^{\prime\,\mu} = p_j^\mu + \frac{z}{2} p_{i\,\dot{A}} \bar{\sigma}^{\mu\,\dot{A}B} p_{j\,B}.$$

Remark: The momenta $p_i^{\prime\mu}$ and $p_j^{\prime\,\mu}$ are in general complex fourvectors.

Cauchy's residue theorem

Consider the amplitude

$$A(z) \quad = \quad A_n\left(p_1^A, p_1^{\dot{B}}, \ldots, p_i^E(z), p_i^{\dot{F}}, \ldots, p_j^M, p_j^{\dot{N}}(z), \ldots, p_n^Y, p_n^{\dot{Z}}\right),$$

with the shifted spinors

$$\begin{aligned} p_i^E(z) &= p_i^E - z p_j^E, \\ p_j^{\dot{N}}(z) &= p_j^{\dot{N}} + z p_i^{\dot{N}}. \end{aligned}$$

- $A(z)$ is a rational function of z.
- $A(z)$ has only simple poles as a function of z.

Cauchy's residue theorem

- If $A(z)$ vanishes at inifinty, it can be written as

$$A(z) \;=\; \sum_{i,j} \frac{c_{ij}}{z - z_{ij}}$$

- The residues c_{ij} are related to the factorization on particle poles:

$$A(z) \;=\; \sum_{i,j} \sum_{\lambda} \frac{A_L^{\lambda}(z_{ij}) A_R^{-\lambda}(z_{ij})}{P_{ij}(z)}$$

- The physical amplitude is obtained by setting $z = 0$ in the denominator. Therefore

$$A \;=\; \sum_{i,j} \sum_{\lambda} \frac{A_L^{\lambda}(z_{ij}) A_R^{-\lambda}(z_{ij})}{P_{ij}}$$

Vanishing at infinity

Example: Helicity combination (i^+, j^-).

Consider flow of z-dependence in a particular diagram. Most dangerous contributions come from diagrams with only three-gluon-vertices along the path.

- Each three-gluon-vertex gives a factor z.
- Each propagator gives a factor $1/z$.
- For a path made of n propagators we have $n+1$ vertices and the product of propagators and vertices behaves like z for large z.
- The polarization vectors contribute a factor $1/z^2$.

The helicity combination (i^+, j^-) behaves like $1/z$ for $z \to \infty$.

Vanishing at infinity

Holomorphic shift: Up to now we shifted p_i^A and $p_j^{\dot{A}}$, while p_j^A and $p_i^{\dot{A}}$ remain unchanged.

Anti-holomorphic shift: Similar considerations apply, if we shift p_j^A and $p_i^{\dot{A}}$, while p_i^A and $p_j^{\dot{A}}$ remain unchanged.

Vanishing at infinity:

	(i^+, j^-)	(i^+, j^+)	(i^-, j^-)	(i^-, j^+)
holomorphic	yes	yes	yes	—
anti-holomorphic	—	yes	yes	yes

On-shell recursion relations for Born QCD amplitudes

The on-shell recursion relations extend to all Born QCD amplitudes.

- Amplitudes with gluons and one massless quark pair.
- Amplitudes with gluons and several massless quark pairs.
- Amplitudes with gluons and massless and/or massive quarks.

The vanishing at $z \to \infty$ is the essential property.

Schwinn and S.W., hep-ph/0703021

Summary

- The last two years witnessed significant new developments for the calculation of amplitudes.
- On-shell recursion relations express an amplitude with n particles in terms of amplitudes with fewer particles.
- Proven for all amplitudes in QCD.
- Applications to loop amplitudes.
- Applications to gravity.

Back-up slides

- Numerical efficiency of Born amplitudes
- Loops

Comparison for Born amplitudes

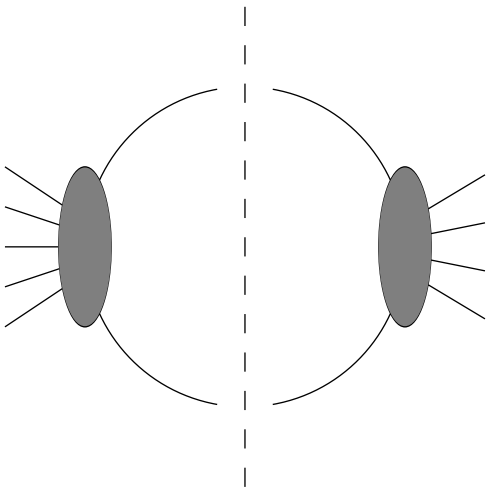
n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00005	0.00023	0.0009	0.003	0.011	0.030	0.09	0.27	0.7
Scalar	0.00008	0.00046	0.0018	0.006	0.019	0.057	0.16	0.4	1
MHV	0.00001	0.00040	0.0042	0.033	0.24	1.77	13	81	—
BCF	0.00001	0.00007	0.0003	0.001	0.006	0.037	0.19	0.97	5.5

CPU time in seconds for the computation of the n gluon amplitude on a standard PC (2 GHz Pentium IV), summed over all helicities.

M. Dinsdale, M. Ternick and S.W., JHEP 0603:056, (hep-ph/0602204);

C. Duhr, S. Höche and F. Maltoni, hep-ph/0607057.

Unitarity method



$$A^{1-loop} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2 + i\varepsilon} \frac{1}{k_2^2 + i\varepsilon} A_L^{tree} A_R^{tree}$$

$+$ cut free pieces

The cut-construction simplifies the calculation of one-loop amplitudes, as cancellations occur already inside A_L^{tree} and A_R^{tree}.

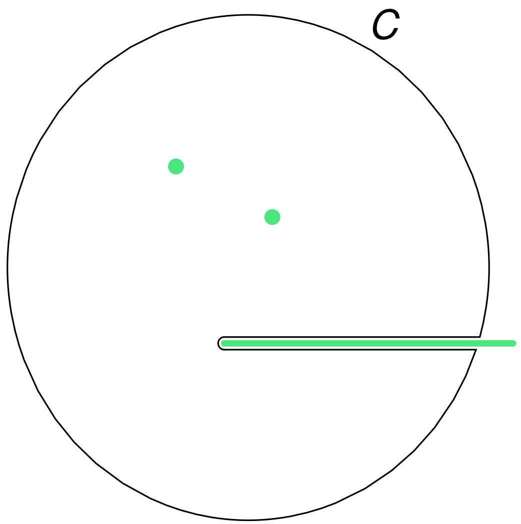
Theorem: One-loop amplitudes in massless supersymmetric gauge theories with no superpotential can be completely constructed from their cuts.

Bern, Dixon, Dunbar and Kosower

Loop amplitudes

Split QCD amplitudes into $N = 4$ and $N = 1$ SUSY pieces and a scalar part.

Loop amplitudes have branch cuts:
Get branch cuts from the unitarity method.
Use recursion relations for the rational pieces.



$$A_n(0) \quad = \quad C_\infty - \sum_{poles} \text{res} \frac{A_n(z)}{z} - \int_{B_0}^{\infty} \frac{dz}{z} \text{Disc}\, A_n(z)$$

Complications: Boundary terms, double poles.

Brandhuber, Spence and Travaglini;

Bern, Dixon, Kosower

One-loop corrections $A_n^{1-loop}(1^-, 2^-, 3^+, \ldots, n^+)$ to adjacent MHV amplitudes have been calculated.

Forde, Kosower

The one-loop six-gluon amplitude

Analytic computation:

Bedford, Berger, bern Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, Cachazo, Dixon, Dunbar, Feng, Forde, Kosower, Mastrolia, Perkins, Spence, Travaglini, Xiao, Yang, Zhu.

$$\mathcal{A}_n = \mathcal{A}_n^{\mathcal{N}=4} - 4\mathcal{A}_n^{\mathcal{N}=1} + \mathcal{A}_n^{\mathcal{N}=0}$$

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0$ (cut)	$\mathcal{N}=0$ (rat)
$--++++$	BDDK (94)	BDDK (94)	BDDK (94)	BDK (94)
$-+-+++$	BDDK (94)	BDDK (94)	BBST (04)	BBDFK (06), XYZ (06)
$-++-++$	BDDK (94)	BDDK (94)	BBST (04)	BBDFK (06), XYZ (06)
$---+++$	BDDK (94)	BDDK (94)	BBDI (05), BFM (06)	BBDFK (06), XYZ (06)
$--+-++$	BDDK (94)	BBDP (05), BBCF (05)	BFM (06)	XYZ (06)
$-+-+-+$	BDDK (94)	BBDP (05), BBCF (05)	BFM (06)	XYZ (06)

Numerical check:

Ellis, Giele, Zanderighi (2006)