A walk on Sunset Boulevard

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Q1: **What functions occur beyond multiple polylogarithms ?**

Q2: **What are the arguments of these functions ?**

in collaboration with
L. Adams, Ch. Bogner, S. Müller-Stach and R. Zayadeh

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the spinor products and masses times two transcendental functions, whose arguments are again algebraic functions of the spinor products and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned} \text{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\ \text{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \end{aligned}$$

Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms (Goncharov 1998):

$$\text{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k) = \sum_{n_1>n_2>...>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot ... \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Iterated integrals

Define the functions G by

$$G(z_1,...,z_k;y) = \int\limits_0^y \frac{dt_1}{t_1 - z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Scaling relation:

$$G(z_1,...,z_k;y) = G(xz_1,...,xz_k;xy)$$

Short hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) = G(\underbrace{0,...,0}_{m_1-1},z_1,...,z_{k-1},\underbrace{0...,0}_{m_k-1},z_k;y)$$

Conversion to multiple polylogarithms:

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1},\frac{1}{x_1 x_2},...,\frac{1}{x_1...x_k};1\right).$$

Differential equations for Feynman integrals

If it is not feasible to compute the integral directly:

Pick one variable t from the set s_{jk} and m_i^2.

1. Find a differential equation for the Feynman integral.

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} I_G(t) \quad = \quad \sum_i q_i(t) I_{G_i}(t)$$

   Inhomogeneous term on the rhs consists of simpler integrals I_{G_i}.

   $p_j(t)$, $q_i(t)$ polynomials in t.

2. Solve the differential equation.

Kotikov; Remiddi, Gehrmann; Laporta; Argeri, Mastrolia; S. Müller-Stach, S.W., R. Zayadeh; Henn; ...

Differential equations: The case of multiple polylogarithms

Suppose the differential operator factorises into linear factors:

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} = \left(a_r(t) \frac{d}{dt} + b_r(t) \right) \dots \left(a_2(t) \frac{d}{dt} + b_2(t) \right) \left(a_1(t) \frac{d}{dt} + b_1(t) \right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the j-th factor by

$$\psi_j(t) = \exp\left(- \int\limits_0^t ds \frac{b_j(s)}{a_j(s)} \right).$$

Full solution given by iterated integrals

$$I_G(t) = C_1 \psi_1(t) + C_2 \psi_1(t) \int\limits_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} + C_3 \psi_1(t) \int\limits_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} \int\limits_0^{t_1} dt_2 \frac{\psi_3(t_2)}{a_2(t_2)\psi_2(t_2)} + \dots$$

Multiple polylogarithms are of this form.

Differential equations: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(t) \frac{d^2}{dt^2} + b_j(t) \frac{d}{dt} + c_j(t)$$

An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[t\left(1-t^2\right)\frac{d^2}{dt^2}+\left(1-3t^2\right)\frac{d}{dt}-t\right]f(t) \quad = \quad 0$$

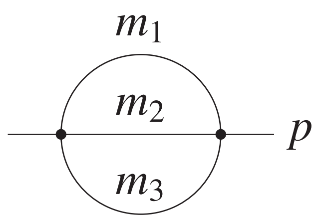
is irreducible.

The solutions of the differential equation are $K(t)$ and $K(\sqrt{1-t^2})$, where $K(t)$ is the complete elliptic integral of the first kind:

$$K(t) \quad = \quad \int_0^1 \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-t^2x^2\right)}}.$$

An example from physics: The two-loop sunset integral

$$S\left(p^2, m_1^2, m_2^2, m_3^2\right) \quad =$$



- Two-loop contribution to the self-energy of massive particles.
- Sub-topology for more complicated diagrams.

Well-studied in the literature:

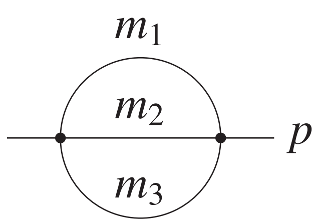
Broadhurst, Fleischer, Tarasov, Bauberger, Berends, Buza, Böhm, Scharf, Weiglein, Caffo, Czyz, Laporta, Remiddi, Groote, Körner, Pivovarov, Bailey, Borwein, Glasser, Adams, Bogner, Müller-Stach, S.W, Zayadeh, Bloch, Vanhove, Tancredi, Pozzorini, Gunia, ...

but still room for further investigations ...

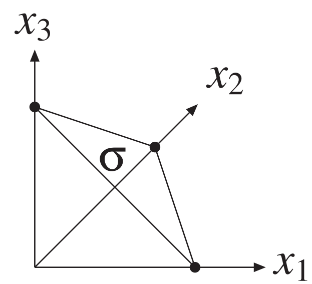
The two-loop sunset integral in two dimensions

In two dimensions: Sunset integral is finite.
Integrand depends only on one graph polynomial.

$$S(t) \quad = $$



$$= \int\limits_{x_j \geq 0} d^3x \, \delta\left(1 - \sum x_j\right) \frac{1}{\mathcal{F}},$$



$$\mathcal{F} \quad = \quad -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right), \qquad t = p^2$$

Algebraic geometry studies the zero sets of polynomials.

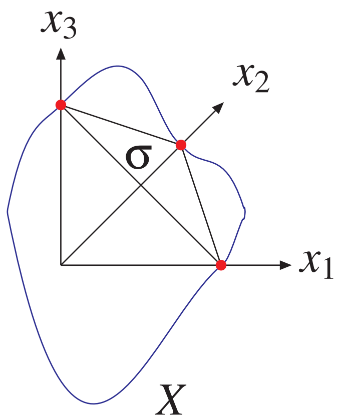
In this case look at the set $\mathcal{F} = 0$.

The two-loop sunset integral

From the point of view of algebraic geometry there are two objects of interest:

- the domain of integration σ,
- the zero set X of $\mathcal{F} = 0$.

X and σ intersect at three points:



The elliptic curve

Algebraic variety X defined by the polynomial in the denominator:

$$-x_1x_2x_3t + \left(x_1m_1^2 + x_2m_2^2 + x_3m_3^2\right)\left(x_1x_2 + x_2x_3 + x_3x_1\right) = 0.$$

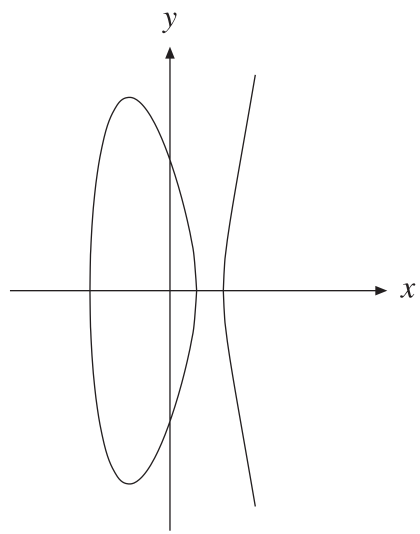
This defines (together with a choice of a rational point as origin) an elliptic curve.

Change of coordinates $\rightarrow$ Weierstrass normal form

$$y^2z - 4x^3 + g_2(t)xz^2 + g_3(t)z^3 = 0.$$

In the chart $z = 1$ this reduces to

$$y^2 - 4x^3 + g_2(t)x + g_3(t) = 0.$$



$y^2 = 4x^3 - 28x + 24$

The curve varies with t.

The second-order differential equation

In two dimensions we have for all values of the masses a second-order differential equation.

The order of the differential equation follows from the fact, that the first cohomology group of an elliptic curve is two dimensional.

$$\left[p_2(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_0(t) \right] S(t) = p_3(t)$$

p_0, p_1, p_2 and p_3 are polynomials in t.

(S. Müller-Stach, S.W., R. Zayadeh, 2011)

Periods of an elliptic curve

In the Weierstrass normal form, factorise the cubic polynomial in x:

$$y^2 = 4\left(x-e_1\right)\left(x-e_2\right)\left(x-e_3\right).$$

Holomorphic one-form is $\frac{dx}{y}$, associated periods are

$$\psi_1\left(t\right) = 2\int\limits_{e_2}^{e_3} \frac{dx}{y}, \qquad \psi_2\left(t\right) = 2\int\limits_{e_1}^{e_3} \frac{dx}{y}.$$

These periods are the solutions of the homogeneous differential equation.

L. Adams, Ch. Bogner, S.W., '13

The full result

- Once the homogeneous solutions are known, variation of the constants yields the full result up to quadrature:
  - Equal mass case: Laporta, Remiddi, ’04
  - Unequal mass case: L. Adams, Ch. Bogner, S.W., ’13

- The full result can be expressed in terms of elliptic dilogarithms:
  - Equal mass case: Bloch, Vanhove, ’13
  - Unequal mass case: L. Adams, Ch. Bogner, S.W., ’14

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}.$$

Elliptic dilogarithm:

$$\mathrm{E}_{2;0}(x;y;q) = \frac{1}{i} \left[\frac{1}{2} \mathrm{Li}_2(x) - \frac{1}{2} \mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right) \right].$$

(Slightly) different definitions of elliptic polylogarithms can be found in the literature

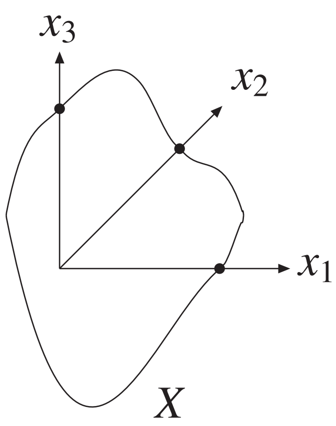
Beilinson ’94, Levin ’97, Brown, Levin ’11, Wildeshaus ’97.

Elliptic curves again

The nome q is given by

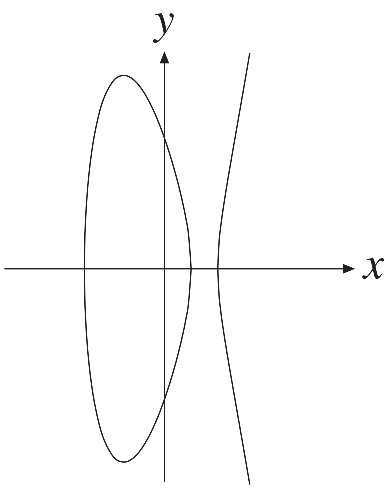
$$q = e^{i\pi\tau} \quad \text{with} \quad \tau = \frac{\psi_2}{\psi_1} = i\frac{K(k')}{K(k)}.$$

Elliptic curve represented by



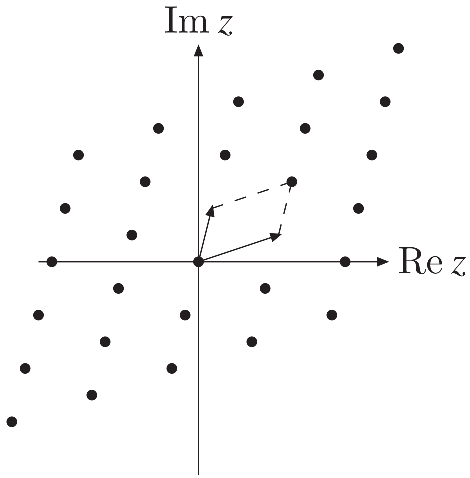
Algebraic variety

$\mathcal{F} = 0$



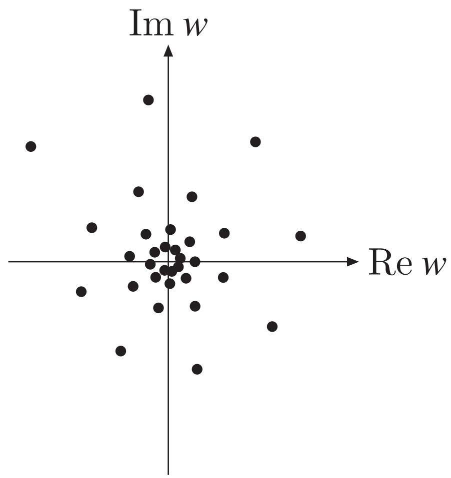
Weierstrass normal form

$y^2 = 4x^3 - g_2 x - g_3$



Torus

$\mathbb{C}/\Lambda$



Jacobi uniformization

$\mathbb{C}^*/q^{2\mathbb{Z}}$

The arguments of the elliptic dilogarithms

Elliptic curve: Cubic curve together with a choice of a rational point as the origin O.

Distinguished points are the points on the intersection of the cubic curve $\mathcal{F} = 0$ with the domain of integration σ:

$$P_1 = [1:0:0], \quad P_2 = [0:1:0], \quad P_3 = [0:0:1].$$

Choose one of these three points as origin and look at the image of the two other points in the Jacobi uniformization $\mathbb{C}^*/q^{2\mathbb{Z}}$ of the elliptic curve. Repeat for the two other choices of the origin. This defines

$$w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}.$$

In other words: $w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}$ are the images of P_1, P_2, P_3 under

$$E_i \longrightarrow \mathrm{WNF} \longrightarrow \mathbb{C}/\Lambda \longrightarrow \mathbb{C}^*/q^{2\mathbb{Z}}.$$

The full result in terms of elliptic dilogarithms

The result for the two-loop sunset integral in two space-time dimensions with arbitrary masses:

$$S = \underbrace{\frac{4}{\left[\left(t-\mu_1^2\right)\left(t-\mu_2^2\right)\left(t-\mu_3^2\right)\left(t-\mu_4^2\right)\right]^{\frac{1}{4}}}}_{\text{algebraic prefactor}} \underbrace{\frac{K\left(k\right)}{\pi}}_{\text{elliptic integral}} \underbrace{\sum_{j=1}^{3} \mathrm{E}_{2;0}\left(w_j;-1;-q\right)}_{\text{elliptic dilogarithms}}$$

t	momentum squared
μ_1, μ_2, μ_3	pseudo-thresholds
μ_4	threshold
$K(k)$	complete elliptic integrals of the first kind
k, q	modulus and nome
w_1, w_2, w_3	points in the Jacobi uniformization

The result in $D = 4 - 2\varepsilon$ dimensions

The result around $D = 4 - 2\varepsilon$ dimensions

Around $D = 4 - 2\varepsilon$ we have the Laurent expansion

$$S(4-2\varepsilon, t) = e^{-2\gamma\varepsilon}\left[\frac{1}{\varepsilon^2} S^{(-2)}(4,t) + \frac{1}{\varepsilon} S^{(-1)}(4,t) + S^{(0)}(4,t) + O(\varepsilon)\right].$$

Around $D = 2 - 2\varepsilon$ we have the Taylor expansion

$$S(2-2\varepsilon, t) = e^{-2\gamma\varepsilon}\left[S^{(0)}(2,t) + \varepsilon S^{(1)}(2,t) + O\left(\varepsilon^2\right)\right].$$

- Pole terms $S^{(-2)}(4,t)$ and $S^{(-1)}(4,t)$ well known, involve only logarithms.
- Dimensional recurrence relations relate $S^{(0)}(4,t)$ to $S^{(0)}(2,t)$ and $S^{(1)}(2,t)$.
- In the equal mass case dependence of $S^{(0)}(4,t)$ on $S^{(1)}(2,t)$ drops out.

Generalisations of the Clausen and Glaisher functions

The Clausen and Glaisher functions are defined by ($x = e^{i\varphi}$):

$$\mathrm{Cl}_n(\varphi) = \begin{cases} \frac{1}{2i}\left[\mathrm{Li}_n(x) - \mathrm{Li}_n\left(x^{-1}\right)\right], \\ \frac{1}{2}\left[\mathrm{Li}_n(x) + \mathrm{Li}_n\left(x^{-1}\right)\right], \end{cases} \qquad \mathrm{Gl}_n(\varphi) = \begin{cases} \frac{1}{2}\left[\mathrm{Li}_n(x) + \mathrm{Li}_n\left(x^{-1}\right)\right], & n \text{ even}, \\ \frac{1}{2i}\left[\mathrm{Li}_n(x) - \mathrm{Li}_n\left(x^{-1}\right)\right], & n \text{ odd}. \end{cases}$$

Elliptic generalisation:

$$\mathrm{E}_{n;m}(x;y;q) =$$

$$= \begin{cases} \frac{1}{i}\left[\frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}\mathrm{Li}_n\left(x^{-1}\right) + \mathrm{ELi}_{n;m}(x;y;q) - \mathrm{ELi}_{n;m}\left(x^{-1};y^{-1};q\right)\right], & n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_n(x) + \frac{1}{2}\mathrm{Li}_n\left(x^{-1}\right) + \mathrm{ELi}_{n;m}(x;y;q) + \mathrm{ELi}_{n;m}\left(x^{-1};y^{-1};q\right), & n+m \text{ odd}. \end{cases}$$

The multi-variable case

In $S^{(1)}(2,t)$ we encounter one multi-variable function.

Sum representation:

$$\mathrm{E}_{0,1;-2,0;4}(x_1,x_2;y_1,y_2;q) =$$

$$\frac{1}{i}\left\{ \left[\mathrm{ELi}_{2;0}(x_1;y_1;q) - \mathrm{ELi}_{2;0}\left(x_1^{-1};y_1^{-1};q\right)\right] \times \frac{1}{2}\left[\mathrm{Li}_1(x_2) + \mathrm{Li}_1\left(x_2^{-1}\right)\right]\right.$$

$$\left. + \sum_{j_1=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{j_2=1}^{\infty}\sum_{k_2=1}^{\infty} \frac{k_1^2}{j_2(j_1k_1+j_2k_2)^2}\left(x_1^{j_1}y_1^{k_1} - x_1^{-j_1}y_1^{-k_1}\right)\left(x_2^{j_2}y_2^{k_2} + x_2^{-j_2}y_2^{-k_2}\right) q^{j_1k_1+j_2k_2}\right\}$$

Integral representation:

$$\mathrm{E}_{0,1;-2,0;4}(x_1,x_2;y_1,y_2;q) = \int_0^q \frac{dq_1}{q_1}\int_0^{q_1}\frac{dq_2}{q_2}\left[\mathrm{E}_{0;-2}(x_1;y_1;q_2) - \mathrm{E}_{0;-2}(x_1;y_1;0)\right]\mathrm{E}_{1;0}(x_2;y_2;q_2)$$

The alphabet

All arguments for the x's and y's are from the set

$$\left\{ w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}, 1, -1 \right\}.$$

The ε^1-term $S^{(1)}(2,t)$ of the expansion around $D = 2 - 2\varepsilon$

Recall: At $O(\varepsilon^0)$ we had

$$S^{(0)}(2,t) \quad = \quad \frac{\psi_1}{\pi} \sum_{j=1}^{3} \mathrm{E}_{2;0}\left(w_j; -1; -q\right)$$

At $O(\varepsilon^1)$ we have now

$$S^{(1)}(2,t) \quad = \quad \frac{\psi_1}{\pi} \left[\sum_{j=1}^{3} \mathrm{E}_{3;1}\left(w_j; -1; -q\right) \; + \; \text{weight 3 terms} \right]$$

with

$$\mathrm{E}_{3;1}\left(x; y; q\right) \quad = \quad \frac{1}{i} \left[\frac{1}{2} \mathrm{Li}_3\left(x\right) - \frac{1}{2} \mathrm{Li}_3\left(x^{-1}\right) + \mathrm{ELi}_{3;1}\left(x; y; q\right) - \mathrm{ELi}_{3;1}\left(x^{-1}; y^{-1}; q\right) \right].$$

Remark: $\mathrm{ELi}_{3;1}$ is of weight 4.

The full result for $S^{(1)}(2,t)$

If you are really curious:

$$S^{(1)}(2,t) = \frac{\psi_1}{\pi}\Bigg\{\Bigg[-\frac{2}{3}\sum_{j=1}^{3}\ln\left(\frac{m_j^2}{\mu^2}\right) - 6\mathrm{E}_{1,0}(-1;1;-q) + \sum_{j=1}^{3}\left(\mathrm{E}_{1;0}(w_j;1;-q) - \frac{1}{3}\mathrm{E}_{1;0}(w_j;-1;-q)\right)\Bigg]\sum_{k=1}^{3}\mathrm{E}_{2;0}(w_k;-1;-q)$$

$$-2\sum_{j=1}^{3}\frac{1}{2i}\Big\{\mathrm{Li}_{2,1}(w_j,1) - \mathrm{Li}_{2,1}\left(w_j^{-1},1\right) + \mathrm{Li}_3(w_j) - \mathrm{Li}_3\left(w_j^{-1}\right) + 3\ln(2)\left[\mathrm{Li}_2(w_j) - \mathrm{Li}_2\left(w_j^{-1}\right)\right]\Big\}$$

$$+\sum_{j=1}^{3}\left[4\mathrm{E}_{0,1;-2,0;4}(w_j,w_j;-1,-1;-q) - 6\mathrm{E}_{0,1;-2,0;4}(w_j,w_j;1,-1;-q) + 6\mathrm{E}_{0,1;-2,0;4}(w_j,-1;-1,1;-q)\right]$$

$$+\sum_{j_1=1}^{3}\sum_{j_2=1}^{3}\left[2\mathrm{E}_{0,1;-2,0;4}(w_{j_1},w_{j_2};1,-1;-q) - \mathrm{E}_{0,1;-2,0;4}(w_{j_1},w_{j_2};-1,-1;-q) - \mathrm{E}_{0,1;-2,0;4}(w_{j_1},w_{j_2};-1,1;-q)\right]$$

$$+\sum_{j=1}^{3}\mathrm{E}_{3;1}(w_j;-1;-q)\Bigg\}$$

Conclusions

Question: What is the next level of sophistication beyond multiple polylogarithms for Feynman integrals?

Answer: Elliptic stuff.

- Algebraic prefactors as before.
- Elliptic integrals generalise the period π.
- Elliptic (multiple) polylogarithms generalise the (multiple) polylogarithms.
- Arguments of the elliptic polylogarithms are points in the Jacobi uniformization of the elliptic curve.
- Weight may raise by two units in the ε-expansion.