Feynman integrals and multiple polylogarithms

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I. **Basic techniques**

II. **Nested sums and iterated integrals**

III. **Multiple Polylogarithms**

IV. **Applications**

The need for precision

Hunting for the Higgs and other yet-to-be-discovered particles requires a better knowledge of the theoretical cross section.

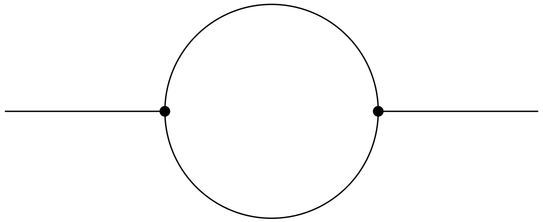
Theoretical predictions are calculated as a power expansion in the coupling. Higher precision is reached by including the next higher term in the perturbative expansion.

State of the art:

- Third or fourth order calculations for a few selected quantities (R-ratio, QCD β-function, anomalous magnetic moment of the muon).
- Fully differential NNLO calculations for a few selected $2 \to 2$ and $2 \to 3$ processes.
- Automated NLO calculations for $2 \to n$ $(n = 4..6, 7)$ processes.

Quantum loop corrections

Loop diagrams are divergent !



$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = \frac{1}{(4\pi)^2} \int_0^\infty dk^2 \frac{1}{k^2} = \frac{1}{(4\pi)^2} \int_0^\infty \frac{dx}{x}$$

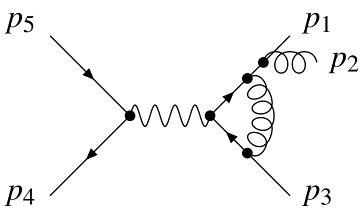
This integral diverges at

- $k^2 \longrightarrow \infty$ (UV-divergence) and at
- $k^2 \longrightarrow 0$ (IR-divergence).

Use dimensional regularization to regulate UV- and IR-divergences.

Feynman rules

A one-loop Feynman diagram contributing to the process $e^+e^- \to qg\bar{q}$:



$$= \quad -\bar{v}(p_4)\gamma^\mu u(p_5)\frac{1}{p_{123}^2}\int \frac{d^D k_1}{(2\pi)^D}\frac{1}{k_2^2}\bar{u}(p_1)\not{\varepsilon}(p_2)\frac{\not{p}_{12}}{p_{12}^2}\gamma_\nu \frac{\not{k}_1}{k_1^2}\gamma_\mu \frac{\not{k}_3}{k_3^2}\gamma^\nu v(p_3)$$

with $p_{12} = p_1 + p_2$, $p_{123} = p_1 + p_2 + p_3$, $k_2 = k_1 - p_{12}$, $k_3 = k_2 - p_3$.
Further $\not{\varepsilon}(p_2) = \gamma_\tau \varepsilon^\tau(p_2)$, where $\varepsilon^\tau(p_2)$ is the polarization vector of the outgoing gluon.
All external momenta are assumed to be massless: $p_i^2 = 0$ for $i = 1..5$.

The loop integral to be calculated reads:

$$\int \frac{d^D k_1}{(2\pi)^D}\frac{k_1^\rho k_3^\sigma}{k_1^2 k_2^2 k_3^2}$$

Standard techniques: Feynman and Schwinger parametrization

Feynman:

$$\prod_{i=1}^{n} \frac{1}{(-A_i)^{\nu_i}} =$$

$$\frac{\Gamma(\nu_1 + \ldots + \nu_n)}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)} \int d^n x \delta\left(1 - \sum_{i=1}^{n} x_i\right) x_1^{\nu_1 - 1} \ldots x_n^{\nu_n - 1} \left(-x_1 A_1 - \ldots - x_n A_n\right)^{-\nu_1 - \ldots - \nu_n}$$

Schwinger:

$$\frac{1}{(-A)^{\nu}} = \frac{1}{\Gamma(\nu)} \int_0^{\infty} dx x^{\nu - 1} \exp(xA)$$

Standard techniques: The cooking recipe

Feynman parameterization.

Shift the loop momentum, such that the denominator becomes purely quadratic in k.

Wick rotation to Euclidean space.

Use spherical coordinates in D dimensions.

The angular integration is trivial, the radial integration can be done with Euler's beta function.

Master formula:

$$\int \frac{d^D k}{\pi^{D/2} i} \frac{(-k^2)^a}{(-k^2 - L)^n} = \frac{\Gamma(D/2+a)}{\Gamma(D/2)} \frac{\Gamma(n-D/2-a)}{\Gamma(n)} (-L)^{-n+D/2+a}$$

This leaves the integrals over the Feynman parameters to be done.

Standard techniques: Tensor integrals

Lorentz symmetry:

$$\int \frac{d^D k}{\pi^{D/2} i} k^\mu k^\nu f(k^2) \quad = \quad \frac{g^{\mu\nu}}{D} \int \frac{d^D k}{\pi^{D/2} i} k^2 f(k^2)$$

From master formula: a factor k^2 in the numerator shifts the dimension $D \to D+2$.

Shifting the loop momentum like in $k' = k - x_1 p_1 - x_2 p_2$ introduces the parameters x_1 or x_2 in the numerator. A Schwinger parameter x in the numerator is equivalent to raising the power of the original propagator by one unit: $\nu \to \nu + 1$.

Summary: Tensor integrals in D dimensions with unit powers of the propagators are equivalent to scalar integrals in $D+2$, $D+4$, ... dimensions and higher powers of the propagators (Tarasov '96).

Standard techniques: Graph polynomials

The general l-loop integral with n propagators:

$$I_G = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}$$

The q_j are linear combinations of the loop momenta k_r and the external momenta.

$$I_G = \frac{\Gamma(\nu - lD/2)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_0^\infty \left(\prod_{j=1}^{n} dx_j x_j^{\nu_j - 1} \right) \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}}, \qquad \nu = \sum_{j=1}^{n} \nu_j.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are graph polynomials, homogeneous of degree l and $l+1$, respectively.

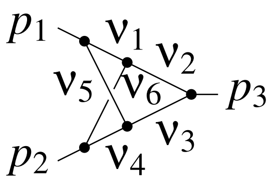
Standard techniques: Graph polynomials

Cutting l lines of a given connected l-loop graph such that it becomes a connected tree graph defines a monomial of degree l. $\mathcal{U}$ is given as the sum over all such monomials.

Cutting one more line leads to two disconnected trees. The corresponding monomials are of degree $l+1$. The function $\mathcal{F}_0$ is the sum over all such monomials times minus the square of the sum of the momenta flowing through the cut lines and

$$\mathcal{F}(x) = \mathcal{F}_0(x) + \mathcal{U}(x) \sum_{j=1}^{n} x_j m_j^2.$$

Example: The massless two-loop non-planar vertex.



$$\begin{aligned} \mathcal{U} &= x_{15}x_{23} + x_{15}x_{46} + x_{23}x_{46}, \\ \mathcal{F} &= (x_1x_3x_4 + x_5x_2x_6 + x_1x_5x_{2346})\left(-p_1^2\right) \\ &\quad + (x_6x_3x_5 + x_4x_1x_2 + x_4x_6x_{1235})\left(-p_2^2\right) \\ &\quad + (x_2x_4x_5 + x_3x_1x_6 + x_2x_3x_{1456})\left(-p_3^2\right). \end{aligned}$$

Standard techniques: Mellin-Barnes

Mellin-Barnes transformation:

$$\left(A_1 + A_2 + \ldots + A_n\right)^{-c} = \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int\limits_{-i\infty}^{i\infty} d\sigma_1 \ldots \int\limits_{-i\infty}^{i\infty} d\sigma_{n-1}$$

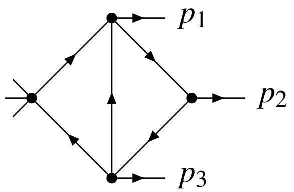
$$\times \Gamma(-\sigma_1) \ldots \Gamma(-\sigma_{n-1}) \Gamma(\sigma_1 + \ldots + \sigma_{n-1} + c) \; A_1^{\sigma_1} \ldots A_{n-1}^{\sigma_{n-1}} A_n^{-\sigma_1 - \ldots - \sigma_{n-1} - c}$$

The contour is such that the poles of $\Gamma(-\sigma)$ are to the right and the poles of $\Gamma(\sigma + c)$ are to the left.

Converts a sum into products and is therefore the "inverse" of Feynman parametrization.

The two-loop C-topology

The result for the C-topology in arbitrary dimensions and with arbitrary powers of the propagators:



$$
= \frac{\Gamma(2m-2\varepsilon-\nu_{1235})\Gamma(1+\nu_{1235}-2m+2\varepsilon)\Gamma(2m-2\varepsilon-\nu_{2345})\Gamma(1+\nu_{2345}-2m+2\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu_5)\Gamma(3m-3\varepsilon-\nu_{12345})} \frac{\Gamma(m-\varepsilon-\nu_5)\Gamma(m-\varepsilon-\nu_{23})}{\Gamma(2m-2\varepsilon-\nu_{235})}
$$

$$
\cdot (-s_{123})^{2m-2\varepsilon-\nu_{12345}} \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \frac{x_1^{i_1}}{i_1!}\frac{x_2^{i_2}}{i_2!} \left[\frac{\Gamma(i_1+\nu_3)\Gamma(i_2+\nu_2)\Gamma(i_1+i_2-2m+2\varepsilon+\nu_{12345})\Gamma(i_1+i_2-m+\varepsilon+\nu_{235})}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+\nu_{23})} \right.
$$

$$
-x_1^{2m-2\varepsilon-\nu_{1235}} \frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+\nu_2)\Gamma(i_1+i_2+\nu_4)\Gamma(i_1+i_2+m-\varepsilon-\nu_1)}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1-2m+2\varepsilon+\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{15})}
$$

$$
-x_2^{2m-2\varepsilon-\nu_{2345}} \frac{\Gamma(i_1+\nu_3)\Gamma(i_2+2m-2\varepsilon-\nu_{345})\Gamma(i_1+i_2+\nu_1)\Gamma(i_1+i_2+m-\varepsilon-\nu_4)}{\Gamma(i_1+1-2m+2\varepsilon+\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{45})}
$$

$$
+x_1^{2m-2\varepsilon-\nu_{1235}} x_2^{2m-2\varepsilon-\nu_{2345}}
$$

$$
\left. \times \frac{\Gamma(i_1+2m-2\varepsilon-\nu_{125})\Gamma(i_2+2m-2\varepsilon-\nu_{345})}{\Gamma(i_1+1+2m-2\varepsilon-\nu_{1235})\Gamma(i_2+1+2m-2\varepsilon-\nu_{2345})} \frac{\Gamma(i_1+i_2+2m-2\varepsilon-\nu_{235})\Gamma(i_1+i_2+3m-3\varepsilon-\nu_{12345})}{\Gamma(i_1+i_2+4m-4\varepsilon-\nu_{12345}-\nu_5)} \right],
$$

with $D=2m-2\varepsilon$, $x_1=(-s_{12})/(-s_{123})$ and $x_2=(-s_{23})/(-s_{123})$.

This sum can be expanded systematically in ε.

Higher transcendental functions

More generally, we get the following types of infinite sums:

- Type A: $$\sum_{i=0}^{\infty} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} x^i$$

  Example: Hypergeometric functions ${}_{J+1}F_J$ (up to prefactors).

- Type B: $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} \frac{\Gamma(j+b_1)...\Gamma(j+b_l)}{\Gamma(j+b'_1)...\Gamma(j+b'_l)} \frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c'_1)...\Gamma(i+j+c'_m)} x^i y^j$$

  Example: First Appell function F_1.

- Type C: $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} \frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c'_1)...\Gamma(i+j+c'_m)} x^i y^j$$

  Example: Kampé de Fériet function S_1.

- Type D: $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{i+j}{j} \frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a'_1)...\Gamma(i+a'_k)} \frac{\Gamma(j+b_1)...\Gamma(j+b_l)}{\Gamma(j+b'_1)...\Gamma(j+b'_l)} \frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c'_1)...\Gamma(i+j+c'_m)} x^i y^j$$

  Example: Second Appell function F_2.

All a, b, c's are of the form "integer $+$ const $\cdot\,\varepsilon$".

Introducing nested sums

- Definition of Z-sums:

$$Z(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{n \geq i_1 > i_2 > ... > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

- Multiple polylogarithms $(n = \infty)$ are a special subset
- Euler-Zagier sums $(x_1 = ... = x_k = 1)$ are a special subset
- The nested sums form a Hopf algebra

Z-sums interpolate between multiple polylogarithms and Euler-Zagier sums.

Special cases

For $n = \infty$ the Z-sums are the multiple polylogarithms of Goncharov:

$$Z(\infty; m_1, ..., m_k; x_1, ..., x_k) \quad = \quad \text{Li}_{m_1,...,m_k}(x_1, ..., x_k)$$

For $x_1 = ... = x_k = 1$ the definition reduces to the Euler-Zagier sums:

$$Z(n; m_1, ..., m_k; 1, ..., 1) \quad = \quad Z_{m_1,...,m_k}(n)$$

For $n = \infty$ and $x_1 = ... = x_k = 1$ the sum is a multiple ζ-value:

$$Z(\infty; m_1, ..., m_k; 1, ..., 1) \quad = \quad \zeta(m_1, ..., m_k)$$

Expansion of Gamma functions

Euler-Zagier sums (or harmonic sums) occur in the expansion for Γ functions: For positive integers n we have

$$\Gamma(n+\varepsilon) = \Gamma(1+\varepsilon)\Gamma(n) \cdot \left(1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)+\varepsilon^3 Z_{111}(n-1)+...+\varepsilon^{n-1} Z_{11...1}(n-1)\right).$$

Z-sums interpolate between Goncharov's multiple polylogarithms and Euler-Zagier sums.

Multiplication

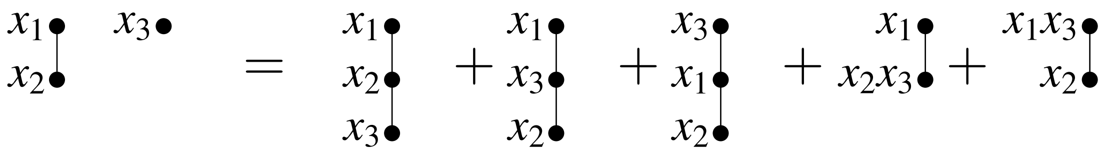
Z-sums obey an algebra:

$$Z(n; m_1, m_2; x_1, x_2) \cdot Z(n; m_3; x_3) =$$

$$= \quad Z(n; m_1, m_2, m_3; x_1, x_2, x_3) + Z(n; m_1, m_3, m_2; x_1, x_3, x_2) + Z(n; m_3, m_1, m_2; x_3, x_1, x_2)$$

$$+ Z(n; m_1, m_2 + m_3; x_1, x_2 x_3) + Z(n; m_1 + m_3, m_2; x_1 x_3, x_2)$$

Pictorial representation:



The multiplication law corresponds to a quasi-shuffle algebra (Hoffman '99),
also called stuffle algebra (Broadhurst)) or mixed shuffle algebra (Guo).

Hopf algebras

The Z-sums form actually a Hopf algebra.

- An algebra has a multiplication $\cdot$ and a unit e.

- A coalgebra has a comultiplication Δ and a counit $\bar{e}$.

- A Hopf algebra is an algebra and a coalgebra at the same time, such that the two structures are compatible with each other.

  In addition, there is an antipode S.

Hopf algebras and renormalization of quantum field theories

Short-distance singularities of the perturbative expansion of quantum field theories require renormalization (Bogoliubov, Parasuik, Hepp, Zimmermann).

The combinatorics involved in the renormalization is governed by a Hopf algebra (Kreimer, Connes).

The model for this Hopf algebra is the Hopf algebra of rooted trees.

Also the Z-sums are described by the Hopf algebra of rooted trees.

Algorithms

Multiplication:

$$Z(n;m_1,...;x_1,...)\cdot Z(n;m'_1,...;x'_1,...)$$

Convolution: Sums involving i and $n-i$

$$\sum_{i=1}^{n-1}\frac{x_1^i}{i^{m_1}}Z(i-1;m_2...;x_2,...)\frac{x_1'^{\,n-i}}{(n-i)^{m'_1}}Z(n-i-1;m'_2,...;x'_2,...)$$

Conjugations:

$$-\sum_{i=1}^{n}\left(\begin{array}{c} n \\ i \end{array}\right)(-1)^i\frac{x_0^i}{i^{m_0}}Z(i;m_1,\ldots,m_k;x_1,\ldots,x_k)$$

Conjugation and convolution: Sums involving binomials and $n-i$

$$-\sum_{i=1}^{n-1}\left(\begin{array}{c} n \\ i \end{array}\right)(-1)^i\frac{x_1^i}{i^{m_1}}Z(i;m_2...;x_2,...)\frac{x_1'^{\,n-i}}{(n-i)^{m'_1}}Z(n-i;m'_2,...;x'_2,...)$$

(Moch, Uwer, S.W., '01)

Hypergeometric Functions

Expansion of hypergeometric functions:

$$_2F_1(a+\varepsilon, b+\varepsilon; c+\varepsilon, x_0) = \sum_{n=0}^{\infty} \frac{(a+\varepsilon)_n (b+\varepsilon)_n}{(c+\varepsilon)_n} \frac{x_0^n}{n!}$$

- Synchronize the Pochhammer symbols using $\Gamma(x+1) = x\Gamma(x)$.
- Expand the Γ-functions in ε. This will introduce Euler-Zagier sums.
- Combine products of Euler-Zagier sums into single sums.
- Read out the harmonic polylogarithms.

Expansion around rational numbers

Up to now we expanded Gamma funktions around integer values. The generalization towards rational numbers p/q introduces the q-th roots of unity:

$$r_q^p = \exp\left(\frac{2\pi i p}{q}\right)$$

Expansion of the Gamma function:

$$\frac{\Gamma\left(n+1-\frac{p}{q}+\varepsilon\right)}{\Gamma\left(1-\frac{p}{q}+\varepsilon\right)} = \frac{\Gamma\left(n+1-\frac{p}{q}\right)}{\Gamma\left(1-\frac{p}{q}\right)} \exp\left(-\frac{1}{q}\sum_{l=0}^{q-1}\left(r_q^l\right)^p \sum_{k=1}^{\infty}\varepsilon^k \frac{(-q)^k}{k} Z(q\cdot n; k; r_q^l)\right).$$

(S.W. '04)

Example

Refinement of sums:

$$\frac{x}{1^2}+\frac{x^4}{4^2}+\frac{x^7}{7^2}+\frac{x^{10}}{10^2}+\ldots = \frac{1}{3}\sum_{n=1}^{\infty}\frac{x^n}{n^2}+\frac{1}{3}e^{\frac{4\pi i}{3}}\sum_{n=1}^{\infty}\frac{\left(xe^{\frac{2\pi i}{3}}\right)^n}{n^2}+\frac{1}{3}e^{\frac{2\pi i}{3}}\sum_{n=1}^{\infty}\frac{\left(xe^{\frac{4\pi i}{3}}\right)^n}{n^2}$$

$$= \frac{1}{3}\mathrm{Li}_2\left(x\right)+\frac{1}{3}e^{\frac{4\pi i}{3}}\mathrm{Li}_2\left(xe^{\frac{2\pi i}{3}}\right)+\frac{1}{3}e^{\frac{2\pi i}{3}}\mathrm{Li}_2\left(xe^{\frac{4\pi i}{3}}\right)$$

Multiple polylogarithms

$$\text{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \ldots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

- Special subsets: Harmonic polylogs, Nielsen polylogs, classical polylogs (Remiddi and Vermaseren, Gehrmann and Remiddi).
- Have also an integral representation.
- Fulfill two Hopf algebras (Moch, Uwer, S.W.).
- Can be evaluated numerically for all complex values of the arguments (Gehrmann and Remiddi, Vollinga and S.W.).

Multiple polylogarithms

The multiple polylogarithms have extensively been studied by Borwein, Bradley, Broadhurst and Lisonek. The multiple polylogarihms contain as subsets the classical polylogarithms

$$\text{Li}_n(x),$$

Nielsen's generalized polylogarithms

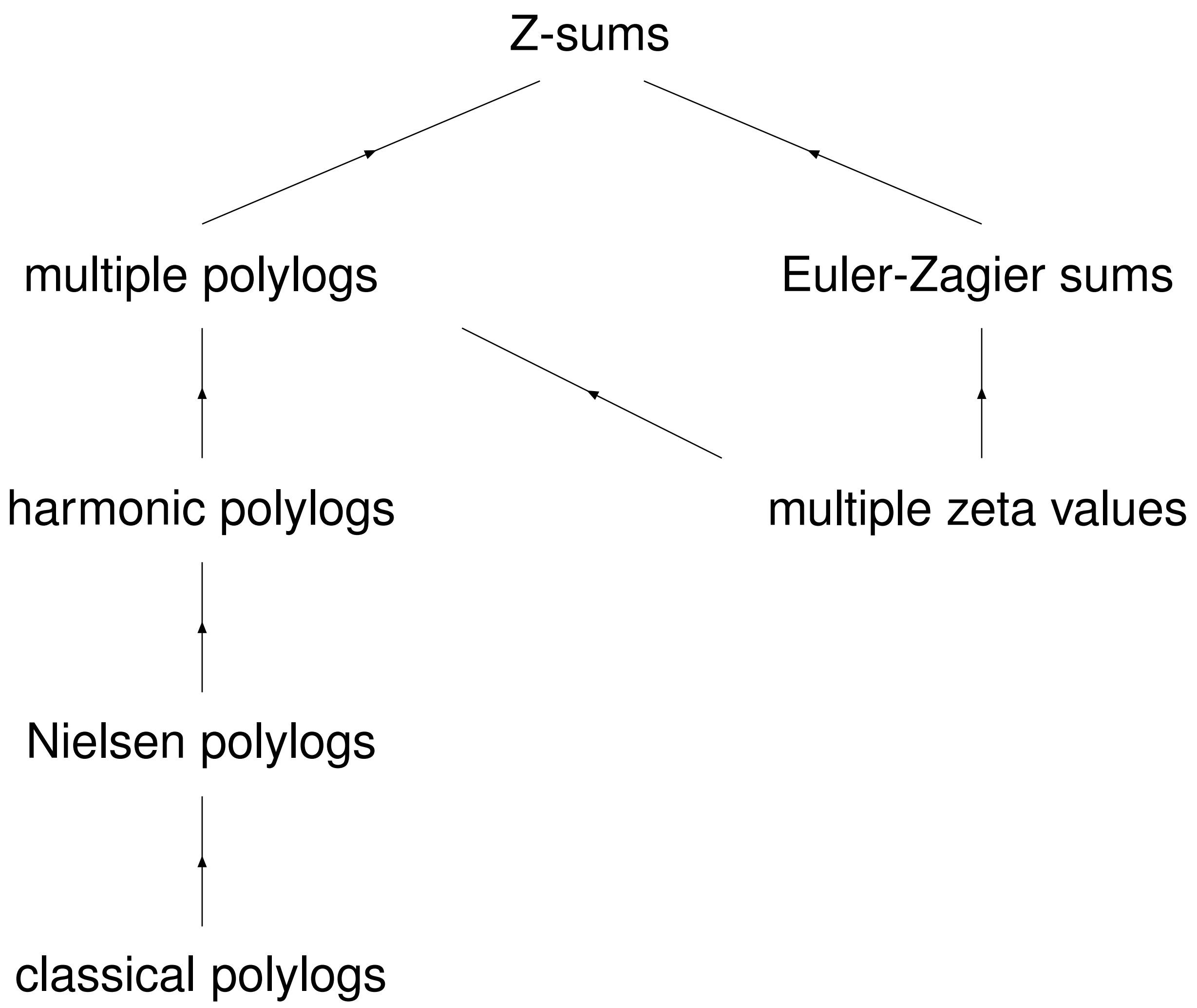
$$S_{n,p}(x) = \text{Li}_{n+1,1,\ldots,1}(x, \underbrace{1,\ldots,1}_{p-1}),$$

the harmonic polylogarithms of Remiddi and Vermaseren

$$H_{m_1,\ldots,m_k}(x) = \text{Li}_{m_1,\ldots,m_k}(x, \underbrace{1,\ldots,1}_{k-1})$$

and the two-dimensional harmonic polylogarithms introduced by Gehrmann and Remiddi.

Inheritance



Iterated integrals

Define the functions G by

$$G(z_1,\ldots,z_k;y) = \int\limits_0^y \frac{dt_1}{t_1-z_1}\int\limits_0^{t_1} \frac{dt_2}{t_2-z_2}\ldots\int\limits_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}.$$

Scaling relation:

$$G(z_1,\ldots,z_k;y) = G(xz_1,\ldots,xz_k;xy)$$

Short hand notation:

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = G(\underbrace{0,\ldots,0}_{m_1-1},z_1,\ldots,z_{k-1},\underbrace{0\ldots,0}_{m_k-1},z_k;y)$$

Conversion to multiple polylogarithms:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\ldots,\frac{1}{x_1\ldots x_k};1\right).$$

Shuffle algebra

The functions $G(z_1, \ldots, z_k; y)$ fulfill a shuffle algebra.

Example:

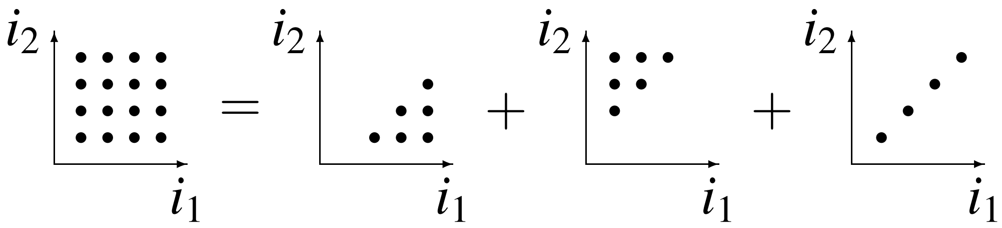
$$G(z_1, z_2; y) G(z_3; y) \quad = \quad G(z_1, z_2, z_3; y) + G(z_1, z_3 z_2; y) + G(z_3, z_1, z_2; y)$$

This algebra is different from the quasi-algebra already encountered and provides the second Hopf algebra for multiple polylogarithms.

Shuffle algebra versus quasi-shuffle algebra

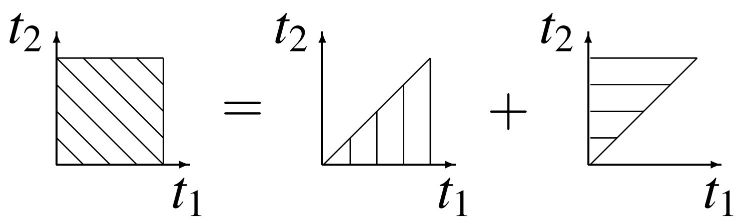
Quasi-shuffle algebra for Z-sums:

$$Z(n;m_1;x_1)Z(n;m_2;x_2) = Z(n;m_1,m_2;x_1,x_2)+Z(n;m_2,m_1;x_2,x_1)+Z(n;m_1+m_2;x_1x_2).$$



Shuffle algebra for G-functions:

$$G(z_1;y)G(z_2;y) = G(z_1,z_2;y)+G(z_2,z_1;y)$$



Partial integration and the antipode

Integration-by-parts identity:

$$
\begin{aligned}
& G(z_1,...,z_k;y) + (-1)^k G(z_k,...,z_1;y) \\
& \quad = \quad G(z_1;y)G(z_2,...,z_k;y) - G(z_2,z_1;y)G(z_3,...,z_k;y) + ... - (-1)^{k-1} G(z_{k-1},...,z_1;y)G(z_k;y)
\end{aligned}
$$

From the Hopf algebra we have the antipode

$$
SG(z_1,...,z_k;y) \quad = \quad (-1)^k G(z_k,...,z_1;y)
$$

Working out the identity for the antipode

$$
\sum_{(G)} S\left(G^{(1)}\right) \cdot G^{(2)} \quad = \quad 0
$$

one recovers the integration-by-parts identity.

The antipode

From the shuffle algebra of the iterated integrals we had:

$$G(z_1,...,z_k;y)+(-1)^k G(z_k,...,z_1;y) = \text{simpler terms}$$

Using the equation for the antipode for Z-sums in the quasi-shuffle algebra:

$$Z(n;m_1,...,m_k;x_1,...,x_k)+(-1)^k Z(n;m_k,..,m_1;x_k,...,x_1) = \text{simpler terms}$$

The equation for the antipode generalizes integration-by-parts identities to cases where no integral representation exists !

Numerical evaluations of multiple polylogarithms

Example: Numerical evaluation of the dilogarithm ('t Hooft, Veltman, Nucl. Phys. B153, (1979), 365)

$$\mathrm{Li}_2(x) = -\int_0^x dt \frac{\ln(1-t)}{t} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Map into region $-1 \leq \mathrm{Re}(x) \leq 1/2$, using

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}(\ln(-x))^2, \qquad \mathrm{Li}_2(x) = -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x).$$

Accelaration using Bernoulli numbers:

$$\mathrm{Li}_2(x) = \sum_{i=0}^{\infty} \frac{B_i}{(i+1)!}(-\ln(1-x))^{i+1},$$

Generalization to multiple polylogarithms, using arbitrary precision arithmetic in C++.

J. Vollinga, S.W., (2004)

Numerical evaluations of multiple polylogarithms

Use the integral representation

$$G_{m_1,\dots,m_k}(z_1,z_2,\dots,z_k;y) = \int\limits_0^y \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{t-z_1} \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{t-z_2} \dots \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{t-z_k}$$

to transform all arguments into a region, where we have a converging power series expansion:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = \sum_{j_1=1}^{\infty} \dots \sum_{j_k=1}^{\infty} \frac{1}{(j_1+\dots+j_k)^{m_1}} \left(\frac{y}{z_1}\right)^{j_1} \frac{1}{(j_2+\dots+j_k)^{m_2}} \left(\frac{y}{z_2}\right)^{j_2} \dots \frac{1}{(j_k)^{m_k}} \left(\frac{y}{z_k}\right)^{j_k}.$$

Use the Hölder convolution to accelerate the convergent series.

(Borwein, Bradley, Broadhurst and Lisonek)

The amplitudes for $e^+e^- \longrightarrow 3$ jets at NNLO

A NNLO calculation of $e^+e^- \longrightarrow 3$ jets requires the following amplitudes:

- Born amplitudes for $e^+e^- \longrightarrow 5$ jets:

  F. Berends, W. Giele and H. Kuijf.

- One-loop amplitudes for $e^+e^- \longrightarrow 4$ jets:

  Z. Bern, L. Dixon, D.A. Kosower and S.W.;

  J. Campbell, N. Glover and D. Miller.

- Two-loop amplitudes for $e^+e^- \longrightarrow 3$ jets:

  L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis and E. Remiddi;

  S. Moch, P. Uwer and S.W.

Results for the two-loop amplitude

The finite part of the coefficient of a spinor structure:

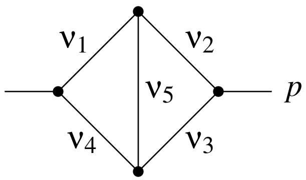
$$c_{12}^{(2),\text{fin}}(x_1,x_2) = N_f N\Bigg(3\frac{\ln(x_1)}{(x_1+x_2)^2}+\frac{1}{4}\frac{\ln(x_2)^2-2\text{Li}_2(1-x_2)}{x_1(1-x_2)}+\frac{1}{12}\frac{\zeta(2)}{(1-x_2)x_1}-\frac{1}{18}\frac{13x_1{}^2+36x_1-10x_1x_2-18x_2+31x_2{}^2}{(x_1+x_2)^2x_1(1-x_2)}\ln(x_2)$$
$$+\frac{x_1{}^2-x_2{}^2-2x_1+4x_2}{(x_1+x_2)^4}\text{R}_1(x_1,x_2)-\frac{1}{12}\frac{\text{R}(x_1,x_2)}{x_1(x_1+x_2)^2}\left[5x_2+42x_1+5-\frac{(1+x_1)^2}{1-x_2}-4\frac{1-3x_1+3x_1{}^2}{1-x_1-x_2}-72\frac{x_1{}^2}{x_1+x_2}\right]+\left[\frac{1}{12}\frac{1}{x_1(1-x_2)}+\frac{6}{(x_1+x_2)^3}\right.$$
$$\left.-\frac{1+2x_1}{x_1(x_1+x_2)^2}\right](\text{Li}_2(1-x_2)-\text{Li}_2(1-x_1))-\frac{1}{(x_1+x_2)x_1}\Bigg)-\frac{1}{2}I\pi N_f N\frac{\ln(x_2)}{x_1(1-x_2)}\,.$$

where $\text{R}(x_1,x_2)$ and $\text{R}_1(x_1,x_2)$ are defined by

$$\text{R}(x_1,x_2) = \left(\frac{1}{2}\ln(x_1)\ln(x_2)-\ln(x_1)\ln(1-x_1)+\frac{1}{2}\zeta(2)-\text{Li}_2(x_1)\right)+(x_1\leftrightarrow x_2)\,.$$

$$\text{R}_1(x_1,x_2) = \Bigg(\ln(x_1)\text{Li}_{1,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)-\frac{1}{2}\zeta(2)\ln(1-x_1-x_2)+\text{Li}_3(x_1+x_2)-\ln(x_1)\text{Li}_2(x_1+x_2)-\frac{1}{2}\ln(x_1)\ln(x_2)\ln(1-x_1-x_2)$$
$$-\text{Li}_{1,2}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)-\text{Li}_{2,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)\Bigg)+(x_1\leftrightarrow x_2)\,.$$

The two-loop two-point function



$$
\begin{aligned}
(1-2\varepsilon)\,\hat{I}^{(2,5)}(2-\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon) = \\
& 6\zeta_3 + 9\zeta_4\varepsilon + 372\zeta_5\varepsilon^2 + \left(915\zeta_6 - 864\zeta_3{}^2\right)\varepsilon^3 \\
& + (18450\zeta_7 - 2592\zeta_4\zeta_3)\,\varepsilon^4 + (50259\zeta_8 - 76680\zeta_5\zeta_3 - 2592\zeta_{6,2})\,\varepsilon^5 \\
& + \left(905368\zeta_9 - 200340\zeta_6\zeta_3 - 130572\zeta_5\zeta_4 + 66384\zeta_3{}^3\right)\varepsilon^6 \\
& + O\left(\varepsilon^7\right).
\end{aligned}
$$

Theorem: Multiple zeta values are sufficient for the Laurent expansion of the two-loop integral $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$, if all powers of the propagators are of the form $\nu_j = n_j + a_j\varepsilon$, where the n_j are positive integers and the a_j are non-negative real numbers.

I. Bierenbaum, S.W., (2003)

Summary

- Systematic algorithms for the calculation of loop integrals based on:
  - nested sums,
  - iterated integrals

- These iterated objects exhibit a rich algebraic structure

- All loop integrals known so far evaluate to multiple polylogarithms