Feynman integrals associated to elliptic curves

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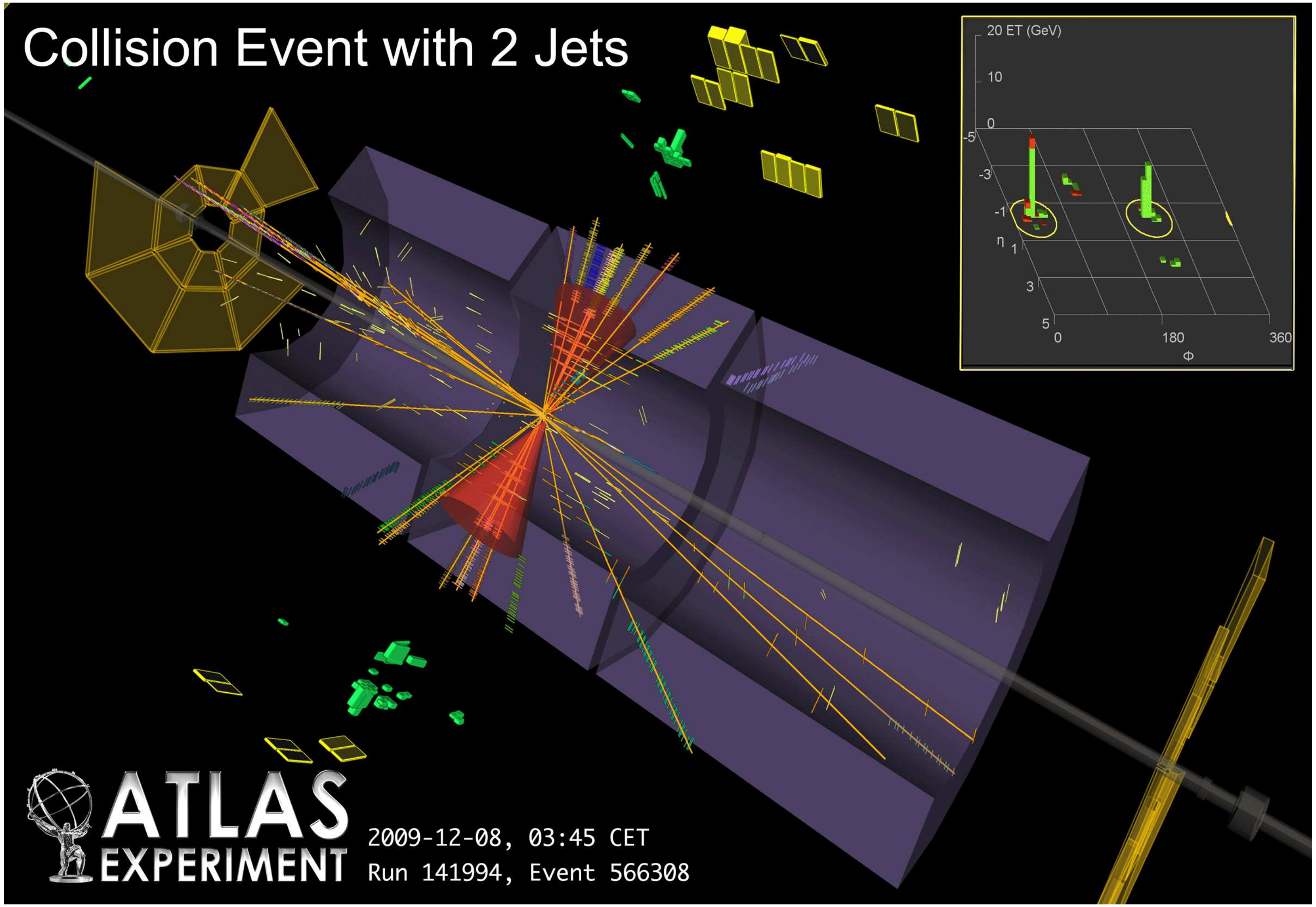
Quantum field theory

- Quantum field theory is the framework for a wide range of physical phenomena.
- At weak coupling we may use perturbation theory.
- Individual terms in the perturbative expansion organised by Feynman diagrams.
- Beyond leading order we get Feynman integrals.

Restrictions for this talk:

- Feynman integrals in momentum space
- Divergent integrals regulated by dimensional regularisation, for example space-time dimension $D = 4 - 2\varepsilon$ with ε infinitesimal.

Experiments in high-energy physics



Examples of applications of perturbative techniques

- High-energy experiments: Precision physics at the Large Hadron Collider (LHC) involving the heaviest particles of the Standard Model (top, Higgs, W- and Z-boson) allows us to probe scales beyond the centre-of-mass energy of $14\,\mathrm{TeV}$.

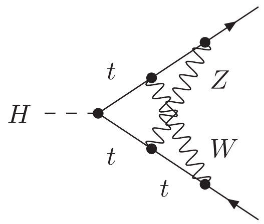
- Low-energy experiments: The Moller experiment at Jefferson lab or the P2 experiment at MESA/Mainz measure the weak mixing angle at low energies to high precision and are therefore sensitive to new physics at TeV-scales complementary to LHC searches.

- Gravitational waves: The inspiral phase of a black-hole merger is described by perturbation theory in gravity.

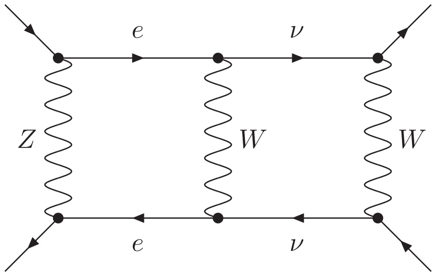
- Spectroscopy: Measurements of the Lamb shift can be used to infer the proton charge radius. The Lamb shift is an effect of quantum electrodynamics.

Examples of Feynman diagrams

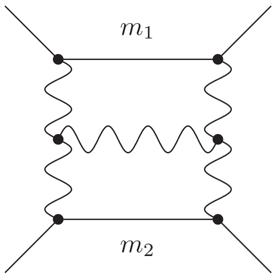
Higgs decay:



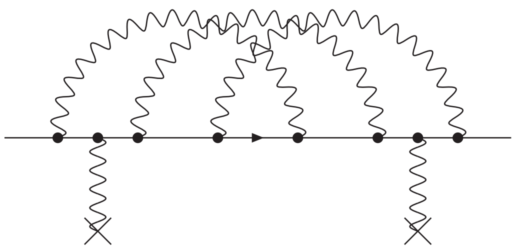
Møller scattering:



Black holes:



Lamb shift:



Highlights of perturbative calculations

- Beta-function in QCD to five loops

  Herzog, Ruijl, Ueda, Vermaseren, Vogt, 17;

  Luthe, Maier, Marquard, Schröder, '17

- Anomalous magnetic moment of the electron in QED to five loops

  Kinoshita et al., '17

  Laporta, '17 (four loops)

- Higgs boson production through gluon fusion to $\mathrm{N^3LO}$ QCD

  Anastasiou, Duhr, Dulat, Herzog, Mistlberger, '15 (large m_t-limit)

State-of-the-art

Despite these impressive calculations, let's ask a different question:

To which loop order can we calculate any Feynman integral?

Here the answer is more modest:

- We can calculate any one-loop integral.
- Already at two-loops there are integrals which are currently not known analytically.

Why?

Rough answer: The complexity of a Feynman integral increases not only with the loop number, but also with the number of kinematic variables.

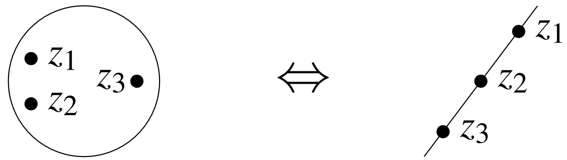
Detailed answer: We understand very well Feynman integrals related to the moduli space $\mathcal{M}_{0,n}$.
From two-loop onwards, we encounter more complicated moduli spaces. The next more complicated moduli space is $\mathcal{M}_{1,n}$.

Moduli spaces

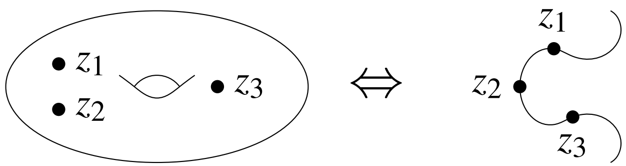
$\mathcal{M}_{g,n}$: Space of isomorphism classes of smooth (complex, algebraic) curves of genus g with n marked points.

Recall:

real surface



complex curve



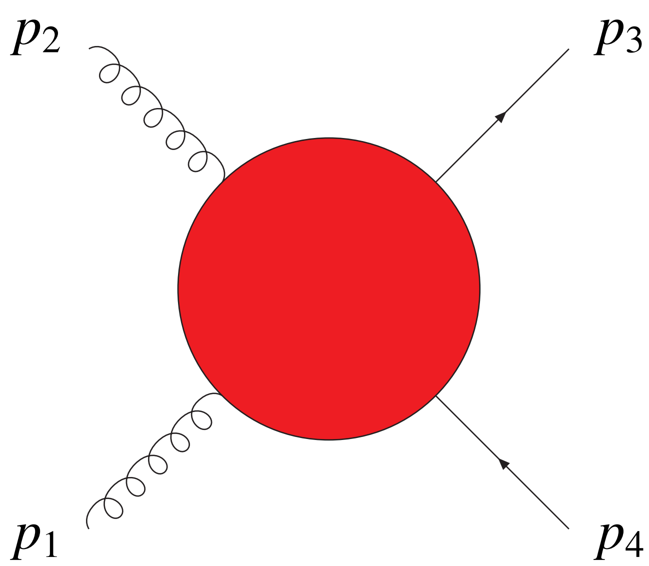
Part 2

No elliptic curves

(Introduction to Feynman integrals)

Scattering amplitudes

For a theoretical description we need to know the scattering amplitude:

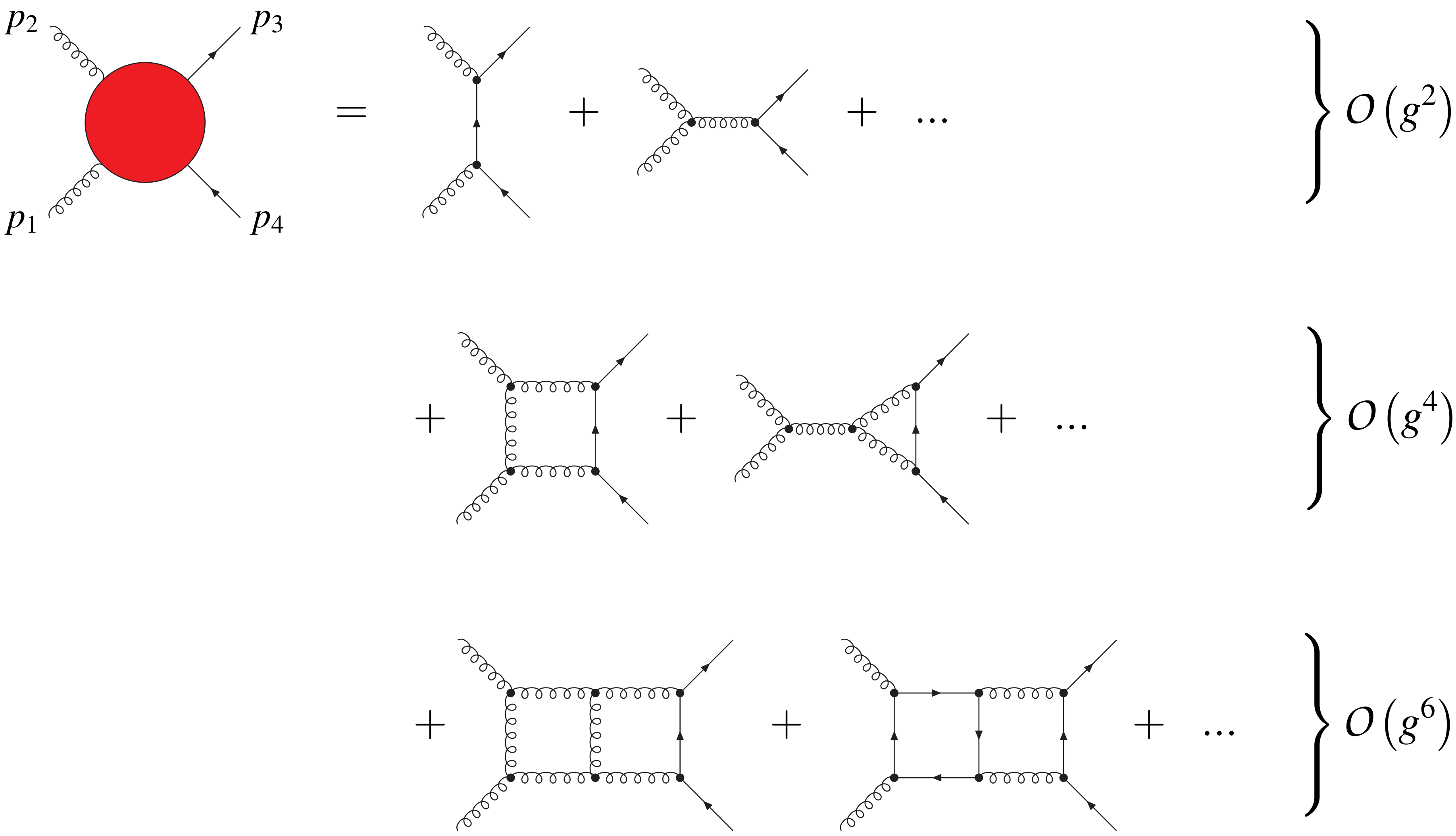


N_{ext} external particles with momenta p_1, ..., $p_{N_{\text{ext}}}$.

Momentum conservation: $p_1 + ... + p_{N_{\text{ext}}} = 0$.

Feynman diagrams

We may compute the scattering amplitude within perturbation theory:



Feynman integrals

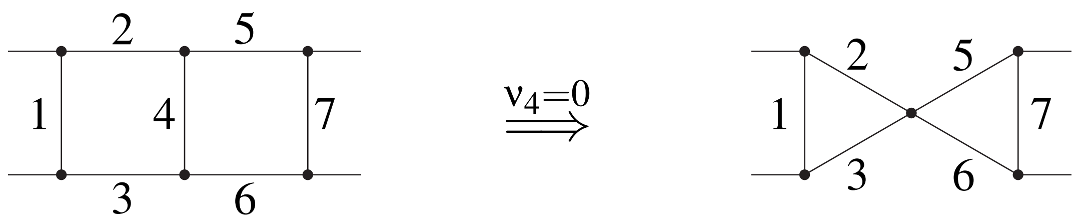
Associate to a Feynman graph G with N_{ext} external lines, n internal lines and l loops the set of Feynman integrals

$$I_{\nu_1 \nu_2 \dots \nu_n} = \int \frac{d^D k_1}{(2\pi)^D} \dots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}},$$

with $\nu_j \in \mathbb{Z}$.

Pinching of propagators

If for some exponent we have $\nu_j = 0$, the corresponding propagator is absent and the topology simplifies:



Integration by parts

Within dimensional regularisation we have for any loop momentum k_i and $v \in \{p_1, ..., p_{N_{\text{ext}}}, k_1, ..., k_l\}$

$$\int \frac{d^D k_1}{(2\pi)^D} ... \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} v^\mu \prod_{j=1}^{n} \frac{1}{\left(q_j^2 - m_j^2\right)^{\nu_j}} = 0.$$

Working out the derivatives leads to relations among integrals with different sets of indices $(\nu_1, ..., \nu_n)$.

This allows us to express most of the integrals in terms of a few master integrals.

Tkachov '81, Chetyrkin '81

Laporta's algorithm

Expressing all integrals in terms of the master integrals requires to solve a rather large linear system of equations.

This system has a block-triangular structure, originating from subtopologies.

Order the integrals by complexity (more propagators $\Rightarrow$ more difficult)

Solve the system bottom-up, re-using the results for the already solved sectors.

Laporta '01

Differential equations

Let x_k be a kinematic variable. Let $I_i \in \{I_1, ..., I_{N_{\mathrm{master}}}\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial x_k} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial x_k} I_i = \sum_{j=1}^{N_{\mathrm{master}}} a_{ij} I_j$$

(Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99)

Differential equations

Let us formalise this:

$I = (I_1, \ldots, I_{N_{\mathrm{master}}})$, set of master integrals,

$x = (x_1, \ldots, x_{N_B})$, set of kinematic variables the master integrals depend on.

We obtain a system of differential equations

$$dI + AI = 0,$$

where $A(\varepsilon, x)$ is a matrix-valued one-form

$$A = \sum_{i=1}^{N_B} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA + A \wedge A = 0 \qquad \text{(flat Gauß-Manin connection)}.$$

Computation of Feynman integrals reduced to solving differential equations!

Simple differential equations

The system of differential equations is particular simple, if A is of the form

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k,$$

where

- C_k is a $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix, whose entries are (rational or integer) numbers,
- the only dependence on ε is given by the explicit prefactor,
- the differential one-forms ω_k have only simple poles.

Henn '13

Chen's iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda = \gamma^* \omega_j.$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) = \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Computation of Feynman integrals reduced to transforming the system of differential equations to a simple form!

Multiple polylogarithms

If all ω_k's are of the form

$$\omega_k \quad = \quad d \ln p_k(x) \, ,$$

where the p_k's are polynomials in the variables x, then (after factorisation of univariate polynomials)

$$f_j \quad = \quad \frac{d\lambda}{\lambda - z_j}$$

and all iterated integrals are multiple polylogarithms:

$$G(z_1, \ldots, z_k; \lambda) \quad = \quad \int_0^{\lambda} \frac{d\lambda_1}{\lambda_1 - z_1} \int_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \ldots \int_0^{\lambda_{k-1}} \frac{d\lambda_k}{\lambda_k - z_k}$$

Example

Let us consider a simple example: One integral I in one variable x with boundary condition $I(0) = 1$. Consider the differential equation

$$(d + A) I = 0, \qquad A = -\varepsilon \, d \ln(x-1).$$

Note that

$$d \ln(x-1) = \frac{dx}{x-1}$$

and

$$I(x) = 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + \ldots$$

Transformations

- Change the basis of the master integrals

$$I' = UI,$$

where $U(\varepsilon, x)$ is a $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix. The new connection matrix is

$$A' = UAU^{-1} + UdU^{-1}.$$

- Perform a coordinate transformation on the base manifold:

$$x_i' = f_i(x), \qquad 1 \leq i \leq N_B.$$

The connection transforms as

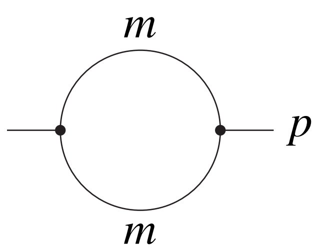
$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Change of coordinates

A change of variables is already required for the one-loop two-loop function, where one encounters ($x = p^2/m^2$)

$$\frac{dx}{\sqrt{-x(4-x)}}.$$

Here, a change of variables in the base manifold



$$x = -\frac{(1-x')^2}{x'}$$

will rationalise the square root and transform

$$\frac{dx}{\sqrt{-x(4-x)}} = \frac{dx'}{x'}$$

Transformations in the case of multiple polylogarithm

- Change the basis of the master integrals

$$I' = UI$$

Systematic algorithms if U is rational in the kinematic variables:

Henn '13; Gehrmann, von Manteuffel, Tancredi, Weihs '14; Argeri et al. '14; Lee '14; Meyer '16; Prausa '17; Gituliar, Magerya '17; Lee, Pomeransky '17;

- Perform a coordinate transformation on the base manifold:

$$x'_i = f_i(x)$$

Algorithms to rationalise square roots:

Becchetti, Bonciani, '17, Besier, van Straten, S.W., '18, Besier, Wasser, S.W., '19.

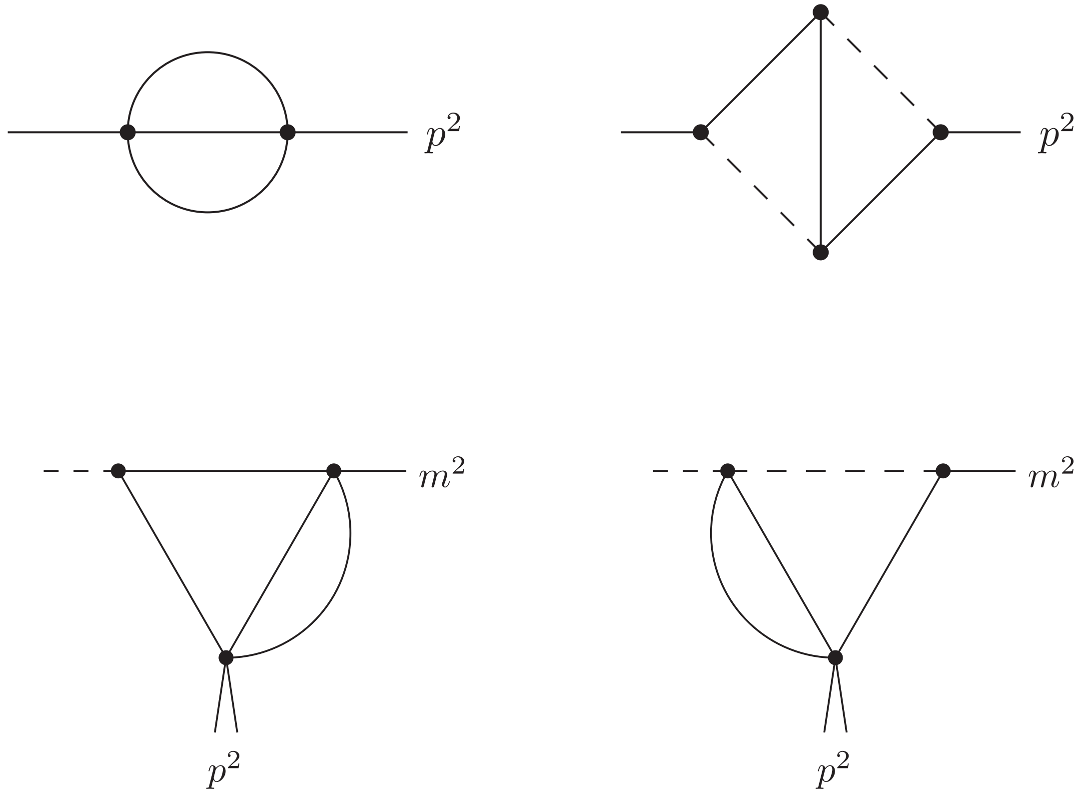
Part 3

One elliptic curve

(Feynman integrals beyond multiple polylogarithms)

Single-scale Feynman integrals beyond multiple polylogarithms

Not all Feynman integrals are expressible in terms of multiple polylogarithms!



The Picard-Fuchs operator

Let I_a be one of the master integrals $\{I_1, ..., I_{N_{\mathrm{master}}}\}$. Choose a path $\gamma : [0,1] \rightarrow M$ and study the integral I_a as a function of the path parameter λ.

Instead of a system of N_{master} first-order differential equations

$$(d+A)\, I \;=\; 0,$$

we may equivalently study a single differential equation of order N_{master}

$$\sum_{j=0}^{N_{\mathrm{master}}} p_j(\lambda) \frac{d^j}{d\lambda^j} I_a \;=\; 0.$$

We may work modulo sub-topologies and ε-corrections:

$$L \;=\; \sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j} \; : \qquad L\, I_a \;=\; 0 \quad \mathrm{mod} \; (\text{sub-topologies, } \varepsilon\text{-corrections})$$

Factorisation of the Picard-Fuchs operator

Suppose the differential operator factorises into linear factors:

$$L = \left(a_r(\lambda) \frac{d}{d\lambda} + b_r(\lambda) \right) \dots \left(a_2(\lambda) \frac{d}{d\lambda} + b_2(\lambda) \right) \left(a_1(\lambda) \frac{d}{d\lambda} + b_1(\lambda) \right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the j-th factor by

$$\psi_j(\lambda) = \exp\left(- \int\limits_0^{\lambda} d\kappa \, \frac{b_j(\kappa)}{a_j(\kappa)} \right).$$

Full solution given by iterated integrals

$$C_1 \psi_1(\lambda) + C_2 \psi_1(\lambda) \int\limits_0^{\lambda} d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1)\psi_1(\lambda_1)} + C_3 \psi_1(\lambda) \int\limits_0^{\lambda} d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1)\psi_1(\lambda_1)} \int\limits_0^{\lambda_1} d\lambda_2 \frac{\psi_3(\lambda_2)}{a_2(\lambda_2)\psi_2(\lambda_2)} + \dots$$

Multiple polylogarithms are of this form.

Picard-Fuchs operator: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(\lambda) \frac{d^2}{d\lambda^2} + b_j(\lambda) \frac{d}{d\lambda} + c_j(\lambda)$$

An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[k\left(1-k^2\right)\frac{d^2}{dk^2} + \left(1-3k^2\right)\frac{d}{dk} - k \right] f(k) \quad = \quad 0$$

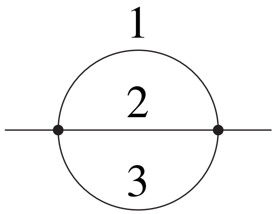
is irreducible.

The solutions of the differential equation are $K(k)$ and $K(\sqrt{1-k^2})$, where $K(k)$ is the complete elliptic integral of the first kind:

$$K(k) \quad = \quad \int\limits_0^1 \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}.$$

An example from physics: The two-loop sunrise integral

$S_{\nu_1\nu_2\nu_3}(D,x) \quad = $



$$x = \frac{p^2}{m^2}$$

Picard-Fuchs operator for $S_{111}(2,x)$:

$$L \quad = \quad x\,(x-1)\,(x-9)\,\frac{d^2}{dx^2} + \left(3x^2 - 20x + 9\right)\frac{d}{dx} + (x-3)$$

(Broadhurst, Fleischer, Tarasov '93)

Irreducible second-order differential operator.

Picard-Fuchs operator for the periods of a family of elliptic curves.

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1x_2x_3x+(x_1+x_2+x_3)(x_1x_2+x_2x_3+x_3x_1) = 0$$

- From the maximal cut:

$$v^2-(u-x)(u-x+4)\left(u^2+2u+1-4x\right) = 0$$

Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

Variables

Recall

$$x = \frac{p^2}{m^2}.$$

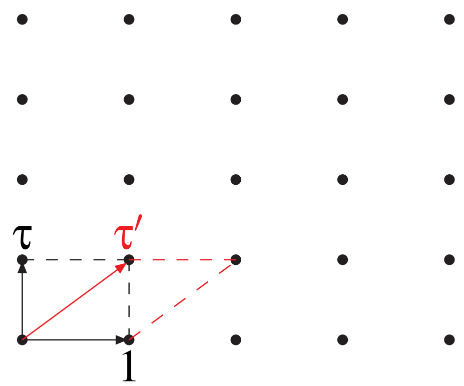
Set

$$\tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{2i\pi\tau}.$$

Change variable from x to τ (or q) (Bloch, Vanhove, '13).

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

The ε-form of the differential equation for the sunrise

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form:

$$I_1 = 4\varepsilon^2 S_{110}(2-2\varepsilon,x), \quad I_2 = -\varepsilon^2 \frac{\pi}{\psi_1} S_{111}(2-2\varepsilon,x), \quad I_3 = \frac{1}{\varepsilon}\frac{1}{2\pi i}\frac{d}{d\tau} I_2 + \frac{1}{24}\left(3x^2-10x-9\right)\frac{\psi_1^2}{\pi^2} I_2.$$

If in addition one makes a (non-algebraic) change of variables from x to τ, one obtains

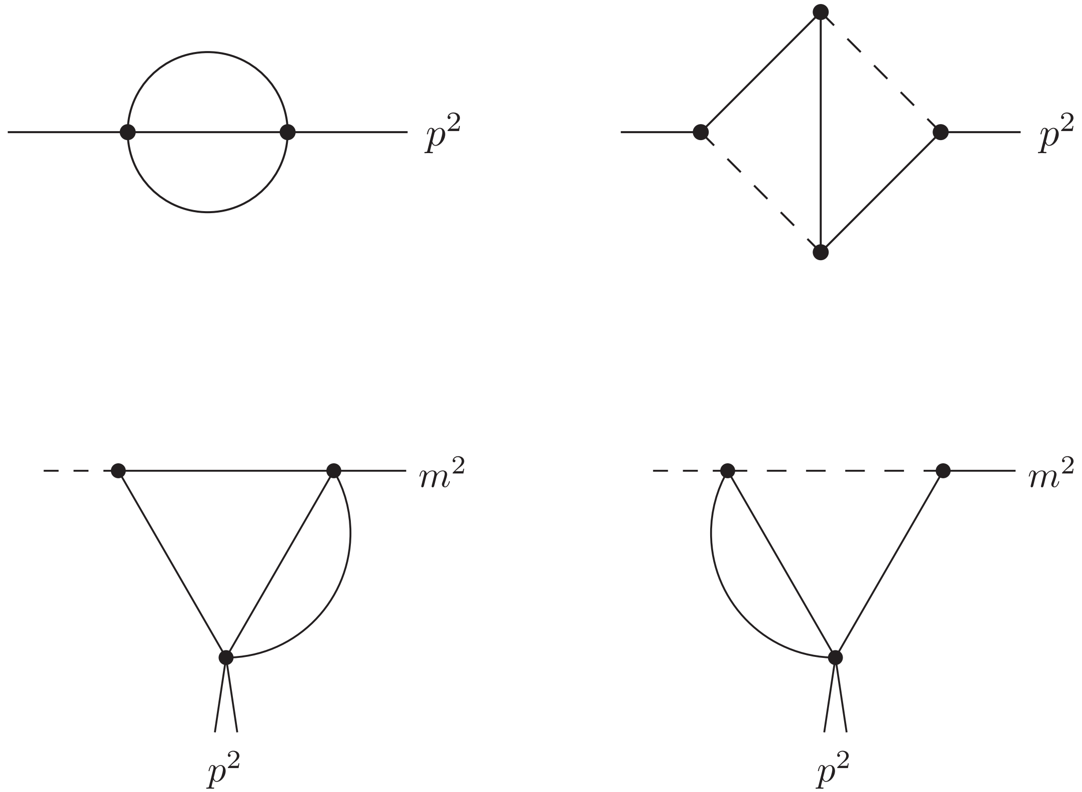
$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k,$$

with $\omega_k = (2\pi)^{2-k} f_k(\tau) \frac{d\tau}{2\pi i}$ and f_k a modular form.

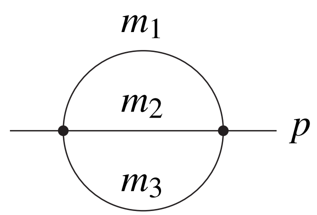
Adams, S.W., ’17, ’18

Feynman integrals evaluating to iterated integrals of modular forms

This applies to a wider class of Feynman integrals:



The unequal mass sunrise integral



There are 7 master integrals. After a redefinition of the basis of master integrals and a change of coordiantes from $(x, y_1, y_2) = (p^2/m_3^2, m_1^2/m_3^2, m_2^2/m_3^2)$ to (τ, z_1, z_2) one finds

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles},$$

where ω_k involves either modular forms or functions appearing in the expansion of the Kronecker function.

Bogner, Müller-Stach, S.W., '19

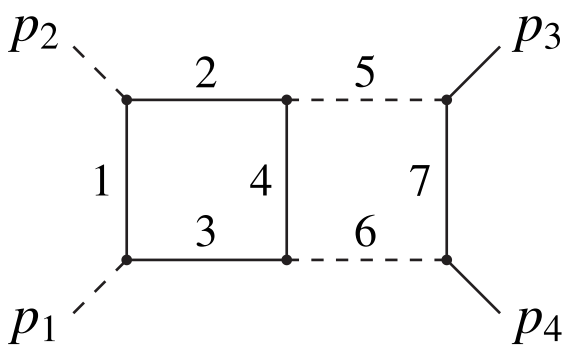
Part 4

Several elliptic curves

(An example from top-pair production)

Kinematics

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}\left(D,\frac{s}{m^2},\frac{t}{m^2}\right)=\left(m^2\right)^{\sum_{j=1}^{7}\nu_j-D}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\prod_{j=1}^{7}\frac{1}{P_j^{\nu_j}},$$



$$p_1^2=p_2^2=0,\qquad p_3^2=p_4^2=m^2,$$

$$s=(p_1+p_2)^2,\qquad t=(p_2+p_3)^2.$$

Picard-Fuchs operator of elliptic curves

- Sunrise integral: An elliptic curve can be obtained either from
  - Feynman graph polynomial
  - maximal cut

  The periods ψ_1, ψ_2 are the solutions of the homogeneous differential equations.

  Adams, Bogner, S.W., '13, '14

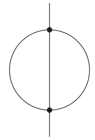
- In general: The maximal cuts are solutions of the homogeneous differential equations.

  Primo, Tancredi, '16

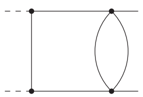
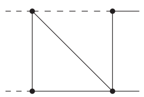
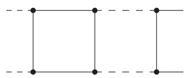
Search for Feynman integrals, whose maximal cuts are periods of an elliptic curve.

Three elliptic curves

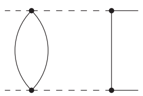
$$E^{(a)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4\right)$$



$$E^{(b)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)$$

$$E^{(c)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+\frac{2m^2\left(s+4t\right)}{\left(s-4m^2\right)}z+\frac{sm^2\left(m^2-4t\right)-4m^2t^2}{s-4m^2}\right)$$



Adams, Chaubey, S.W., '18

Simple differential equations beyond multiple polylogarithms

Can the system of differential equations be brought into the form

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles}$$

for Feynman integrals not evaluating to multiple polylogarithms?

Some explicit examples:

Integral	ε-form	simple poles	comments
all multiple polylogarithms	yes	yes	
equal mass sunrise	yes	yes	$N_B = 1$, 1 elliptic curve
unequal mass sunrise	yes	yes	$N_B = 3$, 1 elliptic curve
topbox	yes	?	$N_B = 2$, 3 elliptic curves

Conclusions

- Feynman integrals important in many areas of physics.

- Feynman integrals evaluating to multiple polylogarithms related to iterated integrals on $\mathcal{M}_{0,n}$.

- Feynman integrals may involve elliptic sectors from two loops onwards.

- There is a class of Feynman integrals evaluating to iterated integrals on $\mathcal{M}_{1,n}$.

- The planar double box integral relavant to $t\bar{t}$-production with a closed top loop depends on two variables and involves several elliptic sub-sectors. More than one elliptic curve occurs.

- We may expect more results in the near future.

Outlook

Computation of Feynman integrals is trivial, as soon as the system of differential equations is transformed to

$$A = \varepsilon \sum_{k=1}^{N_L} C_k \, \omega_k, \qquad \text{with } \omega_k \text{ only simple poles.}$$

This form can be reached for
- many Feynman integrals evaluating to multiple polylogarithms
- a few non-trivial elliptic examples

Open question: Any Feynman integral can be obtained from a system of differential equations of this form.

A constructive proof would gives us an algorithm to compute any Feynman integral.

Back-up slides

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A period ω of the function f is a constant such that for all z:

$$f(z+\omega) = f(z)$$

The set of all periods of f forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\},$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods ω_1 and ω_2.

Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Coordinates on the moduli space

In general: $\dim \mathcal{M}_{g,n} = 3g + n - 3.$

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3.$
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1, \quad z_{n-1} = \infty, \quad z_n = 0$
Coordinates are $(z_1, ..., z_{n-3})$

Genus 1: $\dim \mathcal{M}_{1,n} = n.$
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, z_1, ..., z_{n-1})$

In particular:

$\dim \mathcal{M}_{1,1} = 1$ with coordinate τ, (equal mass sunrise)
$\dim \mathcal{M}_{1,3} = 3$ with coordinates τ, z_1, z_2, (unequal mass sunrise).

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at $i\infty$.

Simple poles at $\tau = i\infty$

A modular form $f_k(\tau)$ is by definition holomorphic at the cusp and has a q-expansion

$$f_k(\tau) = a_0 + a_1 q + a_2 q^2 + ..., \qquad q = \exp(2\pi i \tau)$$

The transformation $q = \exp(2\pi i \tau)$ transforms the point $\tau = i\infty$ to $q = 0$ and we have

$$2\pi i f_k(\tau) d\tau = \frac{dq}{q} \left(a_0 + a_1 q + a_2 q^2 + ... \right).$$

Thus a modular form non-vanishing at the cusp $\tau = i\infty$ has a simple pole at $q = 0$.

The Kronecker function

$$F(z,\alpha,\tau) \quad = \quad \pi\theta_1'(0,q)\frac{\theta_1(\pi(z+\alpha),q)}{\theta_1(\pi z,q)\theta_1(\pi\alpha,q)} = \frac{1}{\alpha}\sum_{k=0}^{\infty} g^{(k)}(z,\tau)\alpha^k, \qquad q = e^{i\pi\tau}$$

Properties of $g^{(k)}(z,\tau)$:

- only simple poles as a function of z
- quasi-periodic as a function of z: Periodic by 1, quasi-periodic by τ.
- almost modular: Nice modular transformation properties only spoiled by divergent Eisenstein series $E_1(z,\tau)$.

Brown, Levin, '11,

Broedel, Duhr, Dulat, Penante, Tancredi, '18

Maximal cuts

Maximal cut: For a Feynman integral

$$I_{\nu_1 \nu_2 \dots \nu_n} = \left(\mu^2\right)^{\nu - lD/2} \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{P_j^{\nu_j}}$$

take the n-fold residue at

$$P_1 = \dots = P_n = 0$$

of the integrand and integrate over the remaining $(lD - n)$ variables along a contour $\mathcal{C}$.