Periods and Feynman integrals

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I. **Motivation for precision calculations**

II. **Standard techniques for Feynman integrals**

III. **Resolution of singularities and periods**

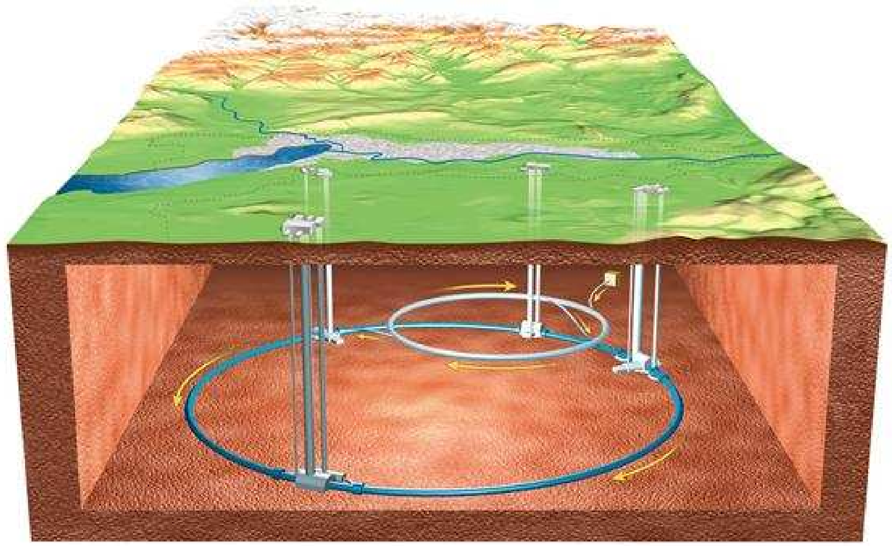
Part I: Motivation for precision calculations

- Experiments in high-energy physics
- Theory: Perturbation theory
- Precision and higher order corrections

Experiments in high-energy physics

Principle: Accelerate two particles, bring them into collision and observe the outcome.

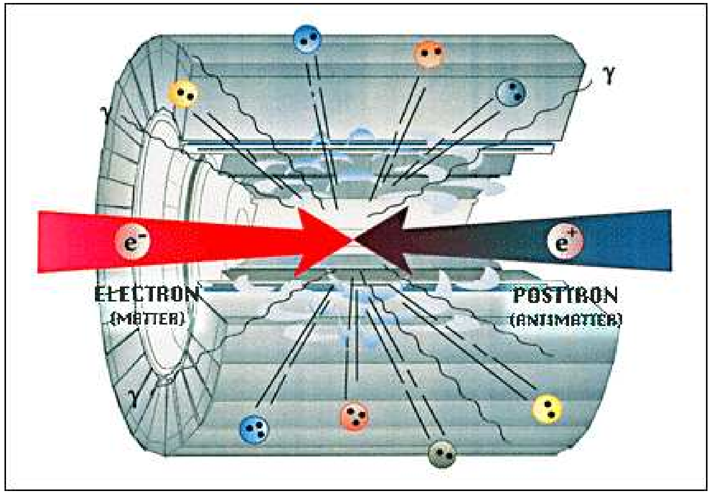
Large Electron Positron Collider (LEP), CERN,
Electron-positron collisions,
Energy: $91 - 210$ GeV.



Tevatron, Fermilab, Chicago,
Proton-antiproton collisions,
Energy: 1.96 TeV.

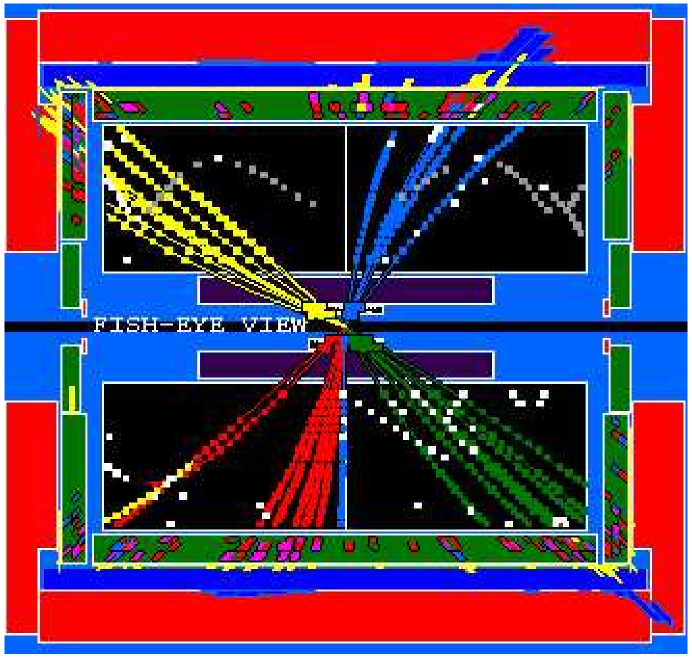
Large hadron collider (LHC), CERN,
Proton-proton collisions,
Energy: 14 TeV.

Experiments in high-energy physics



A schematic view of electron-positron annihilation.

A four-jet event from the Aleph experiment at LEP:



Jets: A bunch of particles moving in the same direction

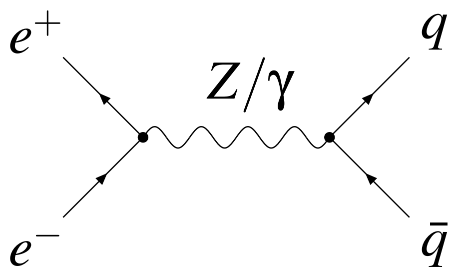
Perturbation theory

Due to the smallness of all coupling constants g, we may compute an observable at high energies reliable in perturbation theory,

$$\sigma = \left(\frac{g}{4\pi}\right)^4 \sigma_{LO} + \left(\frac{g}{4\pi}\right)^6 \sigma_{NLO} + \left(\frac{g}{4\pi}\right)^8 \sigma_{NNLO} + \ldots$$

Cross section related to the square of the scattering amplitude: $\sigma \sim |\mathcal{A}|^2$.
At each order, the amplitude is given as a sum of Feynman diagrams.

A Feynman diagram for electron-positron annihilation:

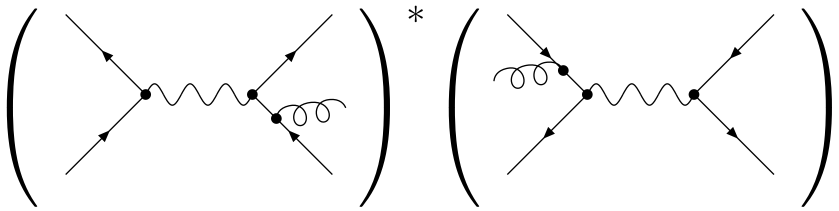


$\sim \quad g^2$

Higher orders in perturbation theory

Higher order contribution to the cross-section $\sigma \sim |\mathcal{A}|^2$:

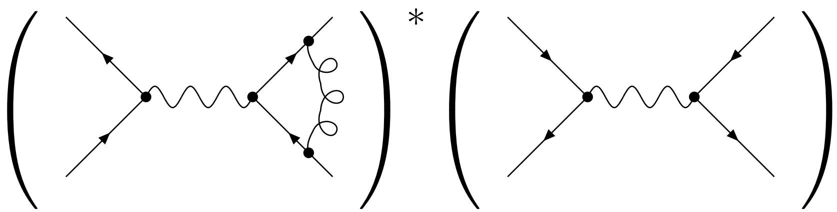
Real emission:



$\sim \quad g^6$

Virtual corrections:

2 Re



$\sim \quad g^6$

Loop diagrams occur in the virtual corrections !

The need for precision

Hunting for the Higgs and other yet-to-be-discovered particles requires a better knowledge of the theoretical cross section.

Theoretical predictions are calculated as a power expansion in the coupling. Higher precision is reached by including the next higher term in the perturbative expansion.

State of the art:

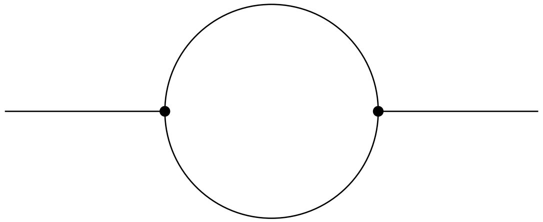
- Third or fourth order calculations for a few selected quantities (R-ratio, QCD β-function, anomalous magnetic moment of the muon).
- Fully differential NNLO calculations for a few selected $2 \to 2$ and $2 \to 3$ processes.
- Automated NLO calculations for $2 \to n$ ($n = 4..6, 7$) processes.

Part II: Standard techniques for Feynman integrals

- Divergences and dimensional regularisation
- The cooking recipe for scalar integrals
- Reduction to scalar integrals

Quantum loop corrections

Loop diagrams are divergent !



$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = \frac{1}{(4\pi)^2} \int_0^\infty dk^2 \frac{1}{k^2} = \frac{1}{(4\pi)^2} \int_0^\infty \frac{dx}{x}$$

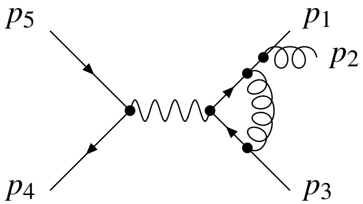
This integral diverges at

- $k^2 \longrightarrow \infty$ (UV-divergence) and at
- $k^2 \longrightarrow 0$ (IR-divergence).

Use dimensional regularization to regulate UV- and IR-divergences.

Feynman rules

A one-loop Feynman diagram contributing to the process $e^+e^- \rightarrow qg\bar{q}$:



$$= \quad -\bar{v}(p_4)\gamma^\mu u(p_5)\frac{1}{p_{123}^2}\int \frac{d^D k_1}{(2\pi)^D}\frac{1}{k_2^2}\bar{u}(p_1)\not{\varepsilon}(p_2)\frac{\not{p}_{12}}{p_{12}^2}\gamma_\nu\frac{\not{k}_1}{k_1^2}\gamma_\mu\frac{\not{k}_3}{k_3^2}\gamma^\nu v(p_3)$$

with $p_{12} = p_1 + p_2$, $p_{123} = p_1 + p_2 + p_3$, $k_2 = k_1 - p_{12}$, $k_3 = k_2 - p_3$.
Further $\not{\varepsilon}(p_2) = \gamma_\tau \varepsilon^\tau(p_2)$, where $\varepsilon^\tau(p_2)$ is the polarization vector of the outgoing gluon.
All external momenta are assumed to be massless: $p_i^2 = 0$ for $i = 1..5$.

The loop integral to be calculated reads:

$$\int \frac{d^D k_1}{(2\pi)^D}\frac{k_1^\rho k_3^\sigma}{k_1^2 k_2^2 k_3^2}$$

Associated scalar loop integral:

$$\int \frac{d^D k_1}{(2\pi)^D}\frac{1}{k_1^2 k_2^2 k_3^2}$$

Standard techniques: Feynman and Schwinger parametrization

Feynman:

$$\prod_{i=1}^{n} \frac{1}{A_i^{\nu_i}} =$$

$$\frac{\Gamma(\nu_1 + \ldots + \nu_n)}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)} \int d^n x \delta\left(1 - \sum_{i=1}^{n} x_i\right) x_1^{\nu_1 - 1} \ldots x_n^{\nu_n - 1} \left(x_1 A_1 + \ldots + x_n A_n\right)^{-\nu_1 - \ldots - \nu_n}$$

Schwinger:

$$\prod_{i=1}^{n} \frac{1}{A_i^{\nu_i}} \quad = \quad \frac{1}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)} \int_0^{\infty} d^n x \, x_1^{\nu_1 - 1} \ldots x_n^{\nu_n - 1} \exp(-x_1 A_1 - \ldots - x_n A_n)$$

The cooking recipe for scalar integrals

Feynman parameterization.

Shift the loop momentum, such that the denominator becomes purely quadratic in k.

Wick rotation to Euclidean space.

Use spherical coordinates in D dimensions.

The angular integration is trivial, the radial integration can be done with Euler's beta function.

This leaves the integrals over the Feynman parameters to be done.

Standard techniques: Tensor integrals

Lorentz symmetry:

$$\int \frac{d^Dk}{\pi^{D/2}i} k^\mu k^\nu f(k^2) \quad = \quad \frac{g^{\mu\nu}}{D} \int \frac{d^Dk}{\pi^{D/2}i} k^2 f(k^2)$$

From master formula: a factor k^2 in the numerator shifts the dimension $D \rightarrow D+2$.

Shifting the loop momentum like in $k' = k - x_1 p_1 - x_2 p_2$ introduces the parameters x_1 or x_2 in the numerator. A Schwinger parameter x in the numerator is equivalent to raising the power of the original propagator by one unit: $\nu \rightarrow \nu + 1$.

Summary: Tensor integrals in D dimensions with unit powers of the propagators are equivalent to scalar integrals in $D+2$, $D+4$, ... dimensions and higher powers of the propagators (Tarasov '96).

Standard techniques: Graph polynomials

The general l-loop integral with n propagators:

$$I_G = \frac{\Gamma(\nu_1)...\Gamma(\nu_n)}{\Gamma(\nu - lD/2)} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}$$

The q_j are linear combinations of the loop momenta k_r and the external momenta.

$$I_G = \int_0^\infty \left(\prod_{j=1}^{n} dx_j x_j^{\nu_j - 1} \right) \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}}, \qquad \nu = \sum_{j=1}^{n} \nu_j.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are graph polynomials, homogeneous of degree l and $l+1$, respectively.

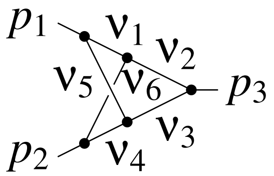
Standard techniques: Graph polynomials

Cutting l lines of a given connected l-loop graph such that it becomes a connected tree graph defines a monomial of degree l. $\mathcal{U}$ is given as the sum over all such monomials.

Cutting one more line leads to two disconnected trees. The corresponding monomials are of degree $l+1$. The function $\mathcal{F}_0$ is the sum over all such monomials times minus the square of the sum of the momenta flowing through the cut lines and

$$\mathcal{F}(x) = \mathcal{F}_0(x) + \mathcal{U}(x) \sum_{j=1}^{n} x_j m_j^2.$$

Example: The massless two-loop non-planar vertex.



$$\begin{aligned}
\mathcal{U} &= x_{15}x_{23} + x_{15}x_{46} + x_{23}x_{46}, \\
\mathcal{F} &= (x_1x_3x_4 + x_5x_2x_6 + x_1x_5x_{2346})\left(-p_1^2\right) \\
&\quad + (x_6x_3x_5 + x_4x_1x_2 + x_4x_6x_{1235})\left(-p_2^2\right) \\
&\quad + (x_2x_4x_5 + x_3x_1x_6 + x_2x_3x_{1456})\left(-p_3^2\right).
\end{aligned}$$

Summary on loop integrals

$$I_G = \int\limits_0^\infty \left(\prod\limits_{j=1}^n dx_j x_j^{\nu_j-1} \right) \delta(1 - \sum\limits_{i=1}^n x_i) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu-lD/2}},$$

$\mathcal{U}$ is a homogeneous polynomial in the Feynman parameters of degree l, positive definite inside the integration region and positive semi-definite on the boundary.

$\mathcal{F}$ is a homogeneous polynomial in the Feynman parameters of degree $l+1$ and depends in addition on the masses m_i^2 and the momenta $(p_{i_1} + ... + p_{i_r})^2$.
In the euclidean region it is also positive definite inside the integration region and positive semi-definite on the boundary.

Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum\limits_{j=-2l}^{\infty} c_j \varepsilon^j.$$

Part III: Resolution of singularities and periods

- Statement of the theorem
- Definition of a period
- Proof of the theorem: Sector decomposition
- Hironaka's polyhedra game

Object of investigation

$$J = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} \left[P_j(x) \right]^{d_j + \varepsilon f_j}.$$

The a's, b's, d's and f's are integers.
The P's are polynomials in the variables x_1, ..., x_n with rational coefficients. The polynomials are required to be positive inside the integration region, but may vanish on the boundaries of the integration region.

The integral J has a Laurent expansion

$$J = \sum_{j=j_0}^{\infty} c_j \varepsilon^j.$$

Theorem: The coefficients c_j of the Laurent expansion of the integral J are periods.

(Bogner, S.W., '07)

Definition of a period

A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

(Kontsevich, Zagier)

Domains defined by polynomial inequalities with rational coefficients are called semi-algebraic sets.

Example:

$$\pi = \iint_{x^2+y^2\leq 1} dx\,dy.$$

The set of all periods is countable.

Periods

Let $G \subset \mathbb{R}^n$ be a semi-algebraic set.
Let $f_1(x)$, $g_1(x)$, $f_2(x)$ and $g_2(x)$ be rational functions with rational coefficients.

If the integrals

$$J = \int_G d^n x \, \{f_1(x) \ln g_1(x) + f_2(x) \ln g_2(x)\}$$

$$K = \int_G d^n x \, f(x) \ln g_1(x) \ln g_2(x)$$

are absolutely convergent, then they are periods.

Prior art: Igusa local zeta functions

Consider the following special case for a Feynman integral:

1. The graph has no external lines or all invariants s_T are zero.

2. All internal masses m_j are equal to 1.

3. All propagators occur with power 1, i.e. $\nu_j = 1$ for all j.

In this case the Feynman parameter integral reduces to

$$I_G = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \, \mathcal{U}^{-D/2}.$$

This integral is a Igusa local zeta function when viewed as a function of $D/2$.
The coefficients of the Laurent expansion are periods.

Belkale and Brosnan, '03

The algorithm of sector decomposition

Back to the general case:

$$J = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} [P_j(x)]^{d_j + \varepsilon f_j}.$$

Algorithm:

Step 0: Convert all polynomials to homogeneous polynomials.
Step 1: Decompose into n primary sectors.
Step 2: Iterate sector decomposition until all polynomials are monomialised.
Step 3: Taylor expansion in the integration variables.
Step 4: Laurent expansion in ε.

Crucial: Termination of step 2.

Hepp; Roth and Denner; Binoth and Heinrich; Bogner and S.W.

Step 0: Convert to homogeneous polynomials

Convert all polynomials to homogeneous polynomials:

$$J = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \left(\prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \right) \prod_{j=1}^{r} [P_j(x)]^{d_j + \varepsilon f_j}.$$

Due to the presence of the delta-function we have

$$1 = x_1 + x_2 + ... + x_n.$$

Can multiply each term in each polynomial P_j by an appropriate power of $x_1 + ... + x_n$.

After step 0 we can assume that all polynomials are homogeneous.

Step 1: Generate primary sectors

Decompose the integral into n primary sectors as in

$$\int_{x_j \geq 0} d^n x \quad = \quad \sum_{l=1}^{n} \int_{x_j \geq 0} d^n x \prod_{i=1, i \neq l}^{n} \theta(x_l \geq x_i).$$

In the l-th primary sector substitute $x_j = x_l x'_j$ for $j \neq l$,
integrate out the variable x_l with the help of the delta-function.

After this step:

$$\int_0^1 d^n x \prod_{i=1}^{n} x_i^{a_i + \varepsilon b_i} \prod_{j=1}^{r} \left[P_j(x)\right]^{d_j + \varepsilon f_j},$$

- integral is now over the unit hypercube,
- the polynomials are positive semi-definite on the unit hypercube,
- zeros may only occur on coordinate subspaces,
- in general the polynomials P_j are no longer homogeneous.

Step 2: Iterate sector decomposition

Decompose the sectors iteratively into sub-sectors until each of the polynomials is monomialised, i.e. of the form

$$P(x) = C x_1^{m_1} \dots x_n^{m_n} \left(1 + P'(x)\right),$$

One iteration: Choose a subset $S = \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots n\}$ according to a strategy. Decompose the k-dimensional hypercube into k sub-sectors according to

$$\int_0^1 d^n x = \sum_{l=1}^{k} \int_0^1 d^n x \prod_{i=1, i \neq l}^{k} \theta\left(x_{\alpha_l} \geq x_{\alpha_i}\right).$$

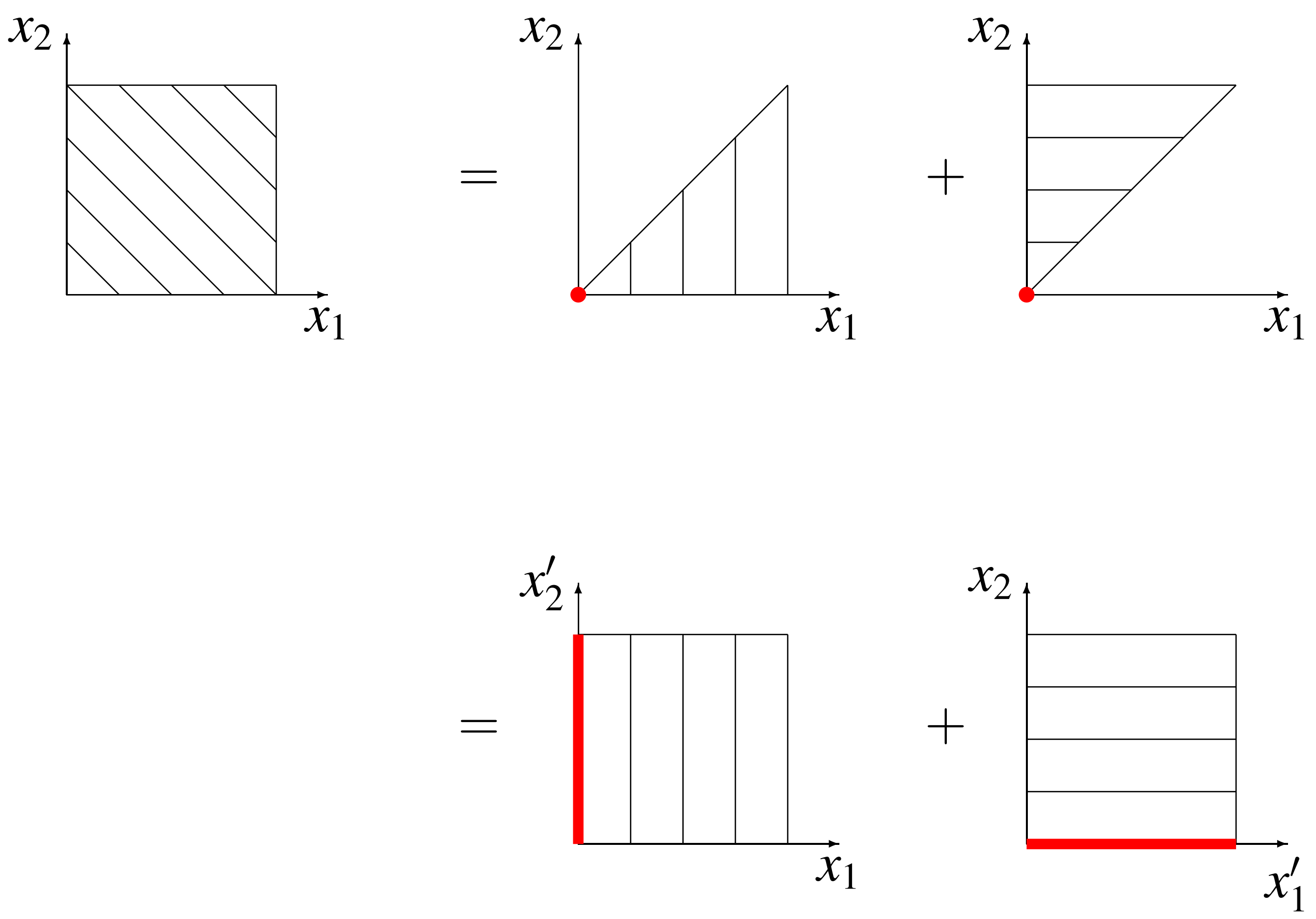
In the l-th sub-sector make for each element of S the substitution

$$x_{\alpha_i} = x_{\alpha_l} x'_{\alpha_i} \quad \text{for } i \neq l.$$

Iterate until all polynomials are monomialised.

Step 2: Blow-ups

Example: $S = \{1, 2\}$.
One iteration blows up the center $x_1 = x_2 = 0$.



Step 2: Strategies

How to choose S ? Have to avoid infinite recursion !

Example:

$$P(x_1, x_2, x_3) = x_1 x_3^2 + x_2^2 + x_2 x_3,$$
$$S = \{1, 2\}$$

First sub-sector: $x_1 = x_1'$, $x_2 = x_1' x_2'$, $x_3 = x_3'$.

$$P(x_1, x_2, x_3) = x_1' x_3'^2 + x_1'^2 x_2'^2 + x_1' x_2' x_3' = x_1' \left(x_3'^2 + x_1' x_2'^2 + x_2' x_3' \right) = x_1' P(x_1', x_3', x_2').$$

The choice $S = \{1, 2\}$ leads to an infinite recursion.

Step 2: Formulation of the problem

We have a product of polynomials

$$\prod_{j=1}^{r} P_j(x)$$

and seek a sequence of blow-ups, such that after a finite number of steps each polynomial is monomialised.

Reformulation:

$$\prod_{j=1}^{r} P_j(x) = 0$$

defines an algebraic variety. We look for a resolution of the singularities of an algebraic variety over a field of characteristic zero by a sequence of blow-ups.

Hironaka, 1964

Step 2: Hironaka's polyhedra game

Two players A and B are given a finite set M of points $m = (m_1, ..., m_n)$ in the first quadrant of $\mathbb{N}^n$. The positive convex hull of the set M is denoted by Δ.

1. Player A chooses a non-empty subset $S \subseteq \{1, ..., n\}$.

2. Player B chooses one element i out of this subset S.

3. All $(m_1, ..., m_n) \in M$ are replaced by new points $(m'_1, ..., m'_n)$:

$$m'_i = \sum_{j \in S} m_j - 1 \quad \text{and} \quad m'_j = m_j, \quad \text{if } j \neq i$$

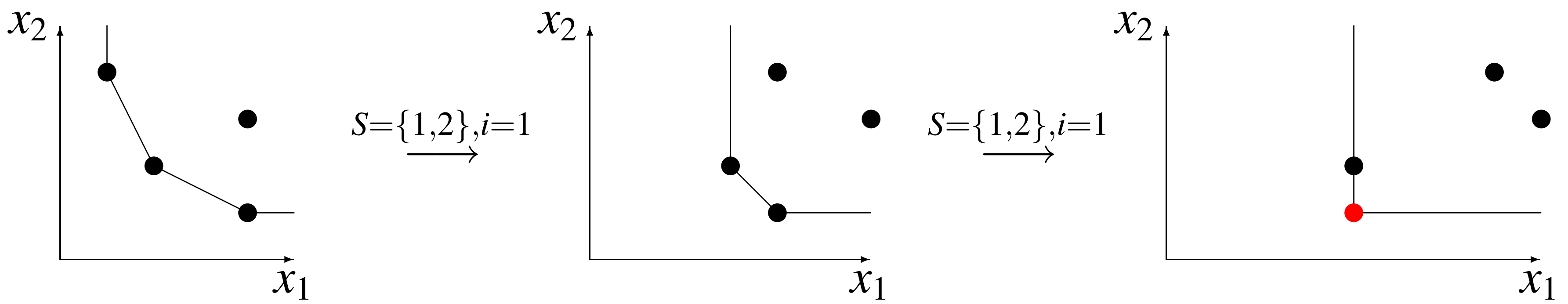
4. Set $M = M'$ and go back to step 1.

Player A wins the game if, after a finite number of moves, the polyhedron Δ is generated by one point, i.e. is of the form

$$\Delta \quad = \quad m + \mathbb{R}^n_+,$$

Step 2: Hironaka's polyhedra game

Example in two dimensions:



Player A wins the game!

Challenge: Find a winning strategy for player A.

Step 2: Winning strategies for the polyhedra game

Common to all winnig strategies is the definition of an invariant and a choice of S based on this invariant, such that any choice of player B will make this invariant decrease.

- Spivakovsky's strategy: The first solution to the polyhedra game.

  Spivakovsky 1983

- Zeillinger's strategy: The simplest one, but also the most inefficient one.

  Zeillinger 2006

- Encinas' and Hauser's strategy: Not restricted to principal ideals, with a lot of effort to make the proof understandable.

  Encinas and Hauser, 2003

See also: Villamayor, Bravo, Bierstone, Milman, ...

Step 3: Taylor expansion in the integration variables

We now have integrals of the form

$$\int_0^1 d^n x \prod_{i=1}^{n} x_i^{a_i+\varepsilon b_i} \prod_{j=1}^{r} \left[1+P'_j(x)\right]^{d_j+\varepsilon f_j},$$

For every x_j with $a_j < 0$ perform a Taylor expansion around $x_j = 0$:

$$\int_0^1 dx_j \, x_j^{a_j+b_j\varepsilon} I\left(x_j\right) = \int_0^1 dx_j \, x_j^{a_j+b_j\varepsilon} \left(\sum_{p=0}^{|a_j|-1} \frac{x_j^p}{p!} I^{(p)} + I^{(R)}(x_j) \right)$$

- The integration in the pole part can be carried out analytically.
- The remainder term is by construction integrable.

Step 3: Taylor expansion in the integration variables

At the end of step 3 we obtain a finite sum of integrals of the form

$$K(\varepsilon) \quad = \quad \frac{1}{g(\varepsilon)} \int_0^1 d^n x \, F(x,\varepsilon),$$

with

$$F(x,\varepsilon) = \sum_{j=1}^{N} f_j(x,\varepsilon), \qquad f_j(x,\varepsilon) = g_j(\varepsilon) \prod_{i=1}^{n} x_i^{a_i^j + \varepsilon b_i} \prod_{k=1}^{r} \left[P_k^j(x) \right]^{d_k^j + \varepsilon f_k}.$$

Here, $g(\varepsilon)$ and $g_j(\varepsilon)$ are polynomials in ε with integer coefficients. $P_k^j(x)$ is a polynomial with rational coefficients, non-vanishing on the unit hypercube. Further we have $a_i^j, b_i, d_i^j, f_i \in \mathbb{Z}$.

Step 3: Taylor expansion in the integration variables

$$\int_0^1 d^n x \, F(x, \varepsilon)$$

is convergent by construction for all ε in a neighbourhood of $\varepsilon = 0$. In one variable this integral is of the form

$$\int_0^1 dx \, x^{\varepsilon b} R(x, \varepsilon),$$

where the function $R(x, \varepsilon)$ does not contain any singularities on the integration domain and is therefore bounded. Therefore the integral is absolutely convergent for all ε with $|\varepsilon| < |1/b|$.

Step 4: Laurent expansion in ε

It remains to expand the integrals in ε: The expansion of the functions $1/g(\varepsilon)$ and $g_j(\varepsilon)$ yields rational numbers, for the other terms we have

$$\begin{aligned} x^{a+b\varepsilon} &= x^a \sum_{k=0}^{\infty} \frac{b^k}{k!} (\ln x)^k \varepsilon^k, \\ [P(x)]^{d+\varepsilon f} &= [P(x)]^d \sum_{k=0}^{\infty} \frac{f^k}{k!} (\ln (P(x)))^k \varepsilon^k. \end{aligned}$$

The integrals over $F_r(x)$ are absolutely convergent: In each variable we have integrals of the form

$$\int_0^1 dx \, (\ln x)^k R_r(x), \quad k \in \mathbb{N}_0,$$

where the function $R_r(x)$ does not contain any singularities on the integration domain and is therefore bounded.

Step 4: Laurent expansion in ε

Summary on sector decomposition:

- Each coefficient of the Laurent expansion is given as a finite sum of integrals.
- These integrals are absolutely convergent.
- All integrals are over the unit hypercube. This is clearly a semi-algebraic set.
- The integrands contain only rational functions with rational coefficients and logarithms thereof.

The coefficients of the Laurent expansion are periods and the theorem is proven.

Summary

- Precision calculations in particle physics
- Standard techniques for Feynman integrals
- A theorem on the Laurent expandion of Feynman integrals:
  - Sector decomposition, blow-ups
  - Hironaka's polyhedra game

  All coefficients are periods !
- The proof is constructive and can be used to compute numerically all coefficients.