New methods for computing helicity amplitudes

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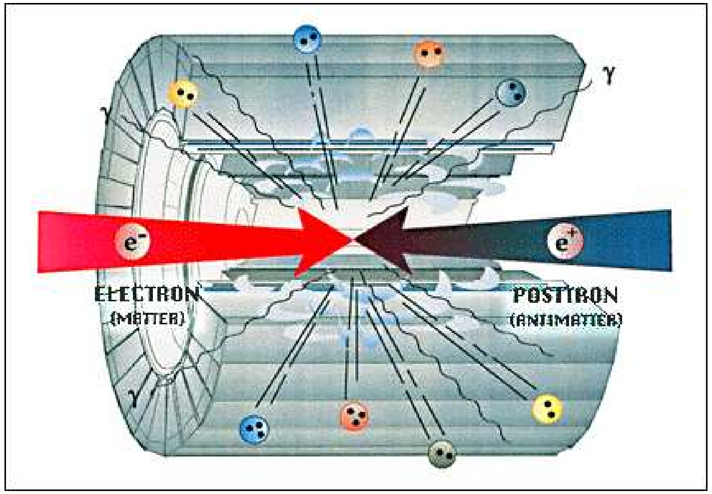
Introduction: **Jet physics**

Part I: **Techniques for many external legs**

Part II: **Twistors, MHV vertices and recurrence relations**

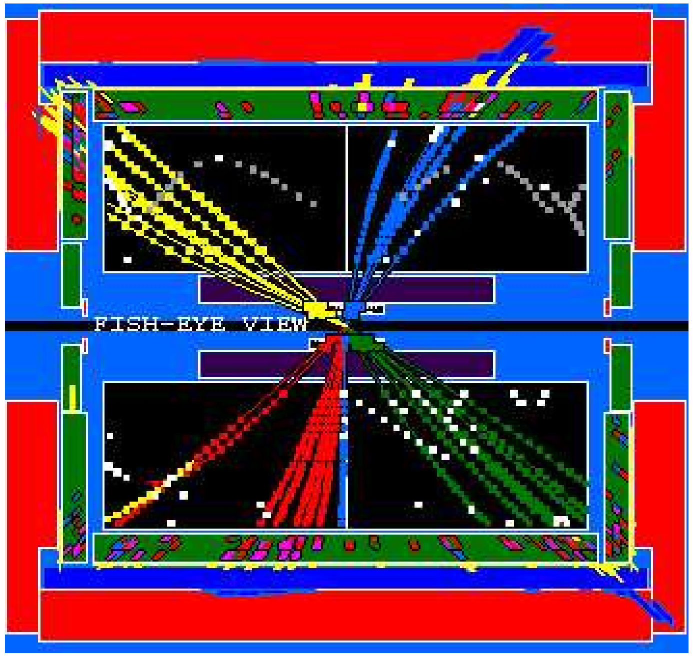
Part III: **Applications**

Jet physics



A schematic view of electron-positron annihilation.

A four-jet event from the Aleph experiment at LEP:



Jets: A bunch of particles moving in the same direction

Jet physics at the LHC

- Jet production: $pp \to$ jets

- Heavy flavour: $pp \to t\bar{t} +$ jets
  $pp \to t\bar{t} + W/Z/H +$ jets

- Single boson: $pp \to W/Z/\gamma +$ jets

- Diboson: $pp \to VV +$ jets

Number of Feynman diagrams contributing to $gg \to ng$ at tree level:

2	4
3	25
4	220
5	2485
6	34300
7	559405
8	10525900

Feynman diagrams are not the method of choice !

Part I : Techniques for many external legs

- Colour decomposition
- Spinor methods
- Supersymmetric relations
- Recurrence relations
- Parke-Taylor formulae
- Unitarity method

Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors carrying the colour structures multiplied by kinematic functions called partial amplitudes.

The partial amplitudes do not contain any colour information and are gauge-invariant. Each partial amplitude has a fixed cyclic order of the external legs.

Examples: The n-gluon amplitude:

$$\mathcal{A}_n(1,2,...,n) \quad = \quad g^{n-2} \sum_{\sigma \in S_n/Z_n} \underbrace{2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)}_{\text{Chan Patton factors}} \; \underbrace{A_n\left(\sigma(1),...,\sigma(n)\right)}_{\text{partial amplitudes}}.$$

P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach,

F. A. Berends and W. Giele,

M. L. Mangano, S. J. Parke, and Z. Xu,

D. Kosower, B.-H. Lee, and V. P. Nair,

Z. Bern and D. A. Kosower.

The spinor helicity method

- Basic objects: Massless two-component Weyl spinors

$$|p\pm\rangle, \qquad \langle p\pm|$$

- Gluon polarization vectors:

$$\varepsilon_\mu^+(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

  q is an arbitrary null reference momentum. Dependency on q drops out in gauge invariant quantities.

- A clever choice of the reference momentum can reduce significantly the number of diagrams which need to be calculated.

Berends, Kleiss, De Causmaecker, Gastmans and Wu; Xu, Zhang and Chang;

Kleiss and Stirling; Gunion and Kunszt

Bra-ket notation versus dotted-undotted indices

Two different notations for the same thing:

$$|p+\rangle = p_B \qquad \langle p+| = p_{\dot{A}}$$

$$|p-\rangle = p^{\dot{B}} \qquad \langle p-| = p^{A}$$

Supersymmetric relations

In an unbroken supersymmetric theory, the supercharge annihilates the vacuum.

$$\langle 0 | [Q, \Phi_1 \Phi_2 ... \Phi_n] | 0 \rangle = 0$$

The supercharge transforms bosons into fermions and vice versa. It relates therefore amplitudes with a pair of fermions to the pure gluon amplitude:

$$A_n^{tree}(q_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, \bar{q}_n^-) = \frac{\langle p_1 - | p_j + \rangle}{\langle p_j - | p_n + \rangle} A_n^{tree}(g_1^+, g_2^+, ..., g_j^-, ..., g_{n-1}^+, g_n^-).$$

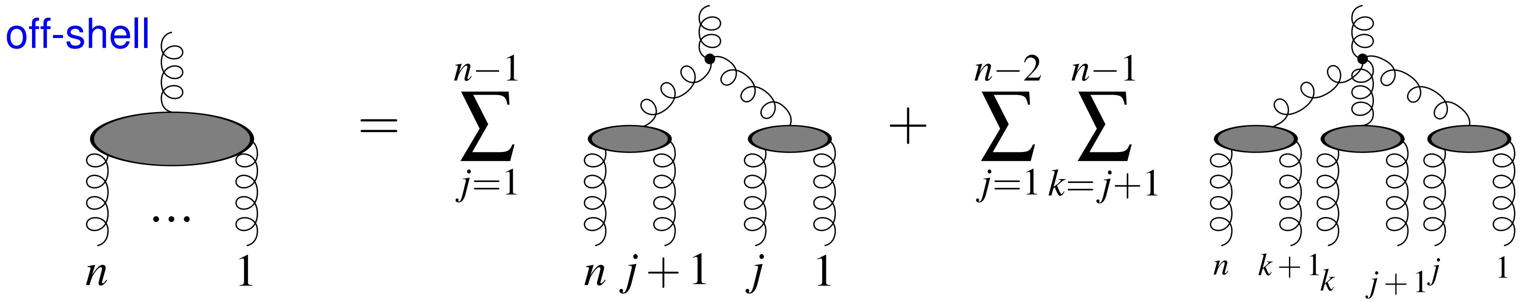
After the colour structure has been stripped off, nothing distinguishes a massless quark from a gluino.

S. J. Parke and T. R. Taylor,

M. T. Grisaru and H. N. Pendleton.

Recurrence relations

Off-shell currents provide an efficient way to calculate amplitudes:



No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele,

D. A. Kosower.

The Parke-Taylor formulae

For specific helicity combinations the amplitudes have a remarkably simple analytic formula or vanish altogether:

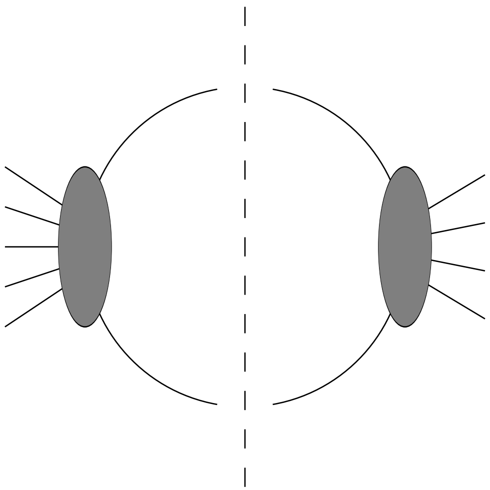
$$
\begin{aligned}
A_n^{tree}(g_1^+, \ldots, g_n^+) &= 0, \\
A_n^{tree}(g_1^+, \ldots, g_j^-, \ldots, g_n^+) &= 0, \\
A_n^{tree}(g_1^+, \ldots, g_j^-, \ldots, g_k^-, \ldots, g_n^+) &= i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk \rangle^4}{\langle 12 \rangle \ldots \langle n1 \rangle}.
\end{aligned}
$$

The n-gluon amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity is called a maximal-helicity violating amplitude (MHV amplitude).

F. A. Berends and W. T. Giele,

S. J. Parke and T. R. Taylor.

Unitarity method



$$A^{1-loop} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k_1^2 + i\varepsilon} \frac{1}{k_2^2 + i\varepsilon} A_L^{tree} A_R^{tree}$$

$+$ cut free pieces

The cut-construction simplifies the calculation of one-loop amplitudes, as cancellations occur already inside A_L^{tree} and A_R^{tree}.

Theorem: One-loop amplitudes in massless supersymmetric gauge theories with no superpotential can be completely constructed from their cuts.

Bern, Dixon, Dunbar and Kosower

Part II : Twistors, MHV vertices and recurrence relations

- Twistor space
- MHV vertices
- BCF recursion relations
- Scalar diagrammatic rules

Twistor space

Each null-vector has a bispinor representation:

$$p^\mu \quad \longrightarrow \quad p_A p_{\dot{B}}$$

Spinors only determined modulo the scaling

$$p_A \longrightarrow \lambda p_A, \qquad p_{\dot{B}} \longrightarrow \frac{1}{\lambda} p_{\dot{B}}.$$

Twistor space: Transform $p_{\dot{B}}$, but not p_A:

$$p_{\dot{A}} \quad \longrightarrow \quad i\frac{\partial}{\partial q^{\dot{A}}},$$

$$-i\frac{\partial}{\partial p^{\dot{A}}} \quad \longrightarrow \quad q_{\dot{A}}.$$

Twistor space continued

In signature $++--$, this transformation can be implemented as a Fourier transformation:

$$A\left(q^{\dot{A}}\right) = \int \frac{d^2 p}{(2\pi)^2} \exp\left(i q^{\dot{A}} p_{\dot{A}}\right) A\left(p_{\dot{A}}\right).$$

In twistor space, the scaling relation reads

$$\left(p_A, q_{\dot{B}}\right) \rightarrow \left(\lambda p_A, \lambda q_{\dot{B}}\right).$$

Therefore twistor space is a three-dimensional projective space.

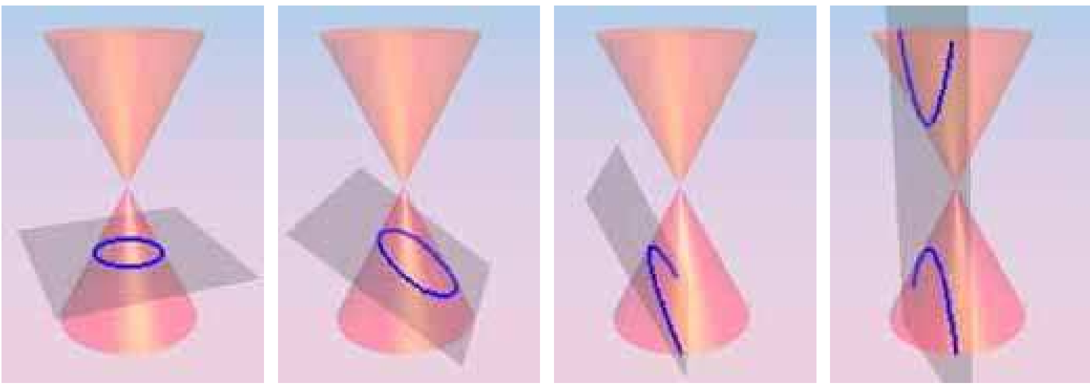
Algebraic curves

Examples of algebraic varieties: The cone is defined by

$$\{(x_1, x_2, x_3)\,; x_1^2 + x_2^2 - x_3^2 = 0\}\,.$$

A conic section is given by

$$\{(x_1, x_2, x_3)\,; x_1^2 + x_2^2 - x_3^2 = 0, ax_1 + bx_2 + cx_3 = 0\}\,.$$



Witten's conjecture

Witten conjectured that the n-gluon amplitude with l-loops is non-zero only if all points lie in twistor space on an algebraic curve of degree d. The degree d of this curve is given by the number of negative helicity gluons plus the number of loops minus one.

E. Witten, Commun. Math. Phys. 252, (2004), 189, (hep-th/0312171)

The CSW construction

Cachazo, Svrček and Witten proposed that the gluonic Born amplitude with an arbitrary helicity configuration can be calculated from diagrams with scalar propagators and new vertices, which are MHV-amplitudes continued off-shell.

$$A_n(1^+,...,j^-,...,k^-,...,n^+) \quad = \quad i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk\rangle^4}{\langle 12\rangle ... \langle n1\rangle}.$$

Off-shell continuation:

$$P \quad = \quad p^\flat + \frac{P^2}{2Pq} q.$$

Propagators are scalars:

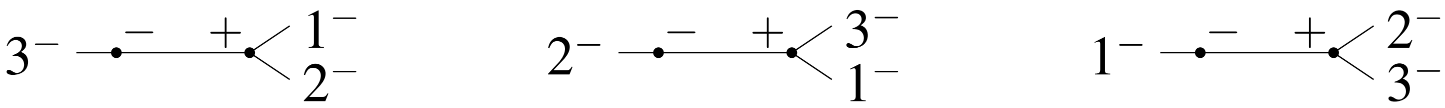
$$\frac{-i}{P^2}$$

Cachazo, Svrček and Witten, JHEP 0409:006, (hep-th/0403047)

Example: Six-gluon amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

The first non-trivial example: The six-gluon amplitude with 3 positive helicity gluons and 3 negative helicity gluons.

One starts with stripped diagrams:



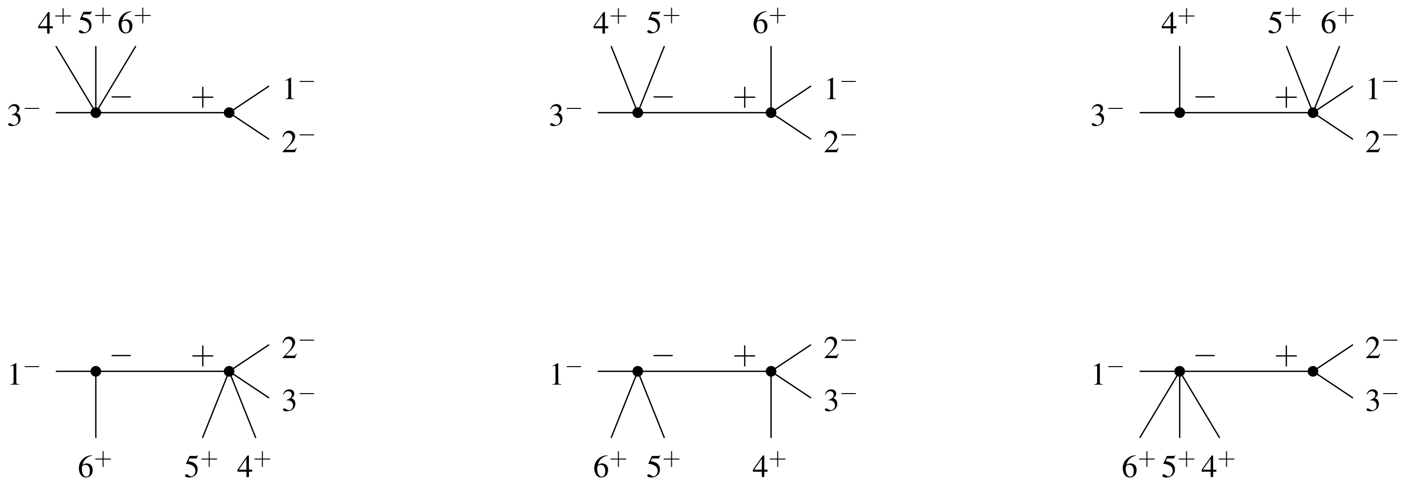
The second diagram will be dressed with all positive helicty gluons inserted between leg 3 and leg 1.

Therefore one MHV vertex with two negative helicity gluons and zero positive helicity gluons remains.

Therefore this diagram does not give a contribution.

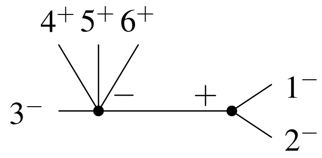
Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

Inserting the gluons with positive helicity:



Example: Six-gluon amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$

The first diagram yields:



$=$

$$\left[i\sqrt{2} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 2 \left(-k_{12}^\flat\right) \rangle \langle \left(-k_{12}^\flat\right) 1 \rangle} \right] \frac{i}{k_{12}^2} \left[i\left(\sqrt{2}\right)^3 \frac{\langle 3 k_{12}^\flat \rangle^4}{\langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 6 k_{12}^\flat \rangle \langle k_{12}^\flat 3 \rangle} \right]$$

Similar for the five other diagrams.

Compare this to

- a brute force approach (220 Feynman diagrams)

- colour-ordered amplitudes (36 diagrams)

The BCF recursion relations

Britto, Cachazo and Feng gave a recursion relation for the calculation of the n-gluon amplitude:

$$A_n\left(p_1, p_2, ..., p_{n-1}^-, p_n^+\right) =$$
$$\sum_{i=1}^{n-3} \sum_{\lambda=+,-} A_{i+2}\left(\hat{p}_n, p_1, p_2, ..., p_i, -\hat{P}_{n,i}^{\lambda}\right) \left(\frac{i}{P_{n,i}^2}\right) A_{n-i}\left(\hat{P}_{n,i}^{-\lambda}, p_{i+1}, ..., p_{n-2}, \hat{p}_{n-1}\right).$$

No off-shell continuation needed. The amplitudes on the r.h.s. are evaluated with shifted momenta.

Britto, Cachazo and Feng, Nucl. Phys. B715, (2005), 499, (hep-th/0412308)

A proof of the BCF recursion relations

Consider the amplitude

$$A(z) = A\left(p_1, ..., p_k(z), ..., p_{n-1}, p_n(z)\right)$$

with shifted momenta

$$\begin{aligned} p_{k,A\dot{B}}(z) &= p_{k,A}\left(p_{k,\dot{B}} - z p_{n,\dot{B}}\right), \\ p_{n,A\dot{B}}(z) &= \left(p_{n,A} + z p_{k,A}\right) p_{n,\dot{B}}. \end{aligned}$$

- $A(z)$ is a rational function of z.
- $A(z)$ has only simple poles as a function of z.

A proof of the BCF recursion relations

- If $A(z)$ vanishes at inifinty, it can be written as

$$A(z) = \sum_{i,j} \frac{c_{ij}}{z - z_{ij}}$$

- The residues c_{ij} are related to the factorization on particle poles:

$$A(z) = \sum_{i,j} \sum_{\lambda} \frac{A_L^{\lambda}(z_{ij}) A_R^{-\lambda}(z_{ij})}{P_{ij}(z)}$$

- The physical amplitude is obtained by setting $z = 0$ in the denominator. Therefore

$$A = \sum_{i,j} \sum_{\lambda} \frac{A_L^{\lambda}(z_{ij}) A_R^{-\lambda}(z_{ij})}{P_{ij}}$$

Britto, Cachazo, Feng and Witten, Phys. Rev. Lett. 94:181602, (2005), (hep-th/0501052),

Draggiotis, Kleiss, Lazopoulos and Papadopoulos, hep-ph/0511288

Axial gauge

Polarisation sum, continued off-shell:

$$\sum_{\lambda=+/-} \varepsilon_\mu^\lambda(k^\flat, q)\varepsilon_\nu^{-\lambda}(k^\flat, q) \quad = \quad -g_{\mu\nu} + 2\frac{k_\mu^\flat q_\nu + q_\mu k_\nu^\flat}{2kq}.$$

The gluon propagator in the axial gauge is given by

$$\frac{i}{k^2} d_{\mu\nu} \quad = \quad \frac{i}{k^2}\left(-g_{\mu\nu} + 2\frac{k_\mu q_\nu + q_\mu k_\nu}{2kq}\right) = \frac{i}{k^2}\left(\varepsilon_\mu^+ \varepsilon_\nu^- + \varepsilon_\mu^- \varepsilon_\nu^+ + \varepsilon_\mu^0 \varepsilon_\nu^0\right),$$

where we introduced an unphysical polarisation

$$\varepsilon_\mu^0(k, q) \quad = \quad 2\frac{\sqrt{k^2}}{2kq} q_\mu.$$

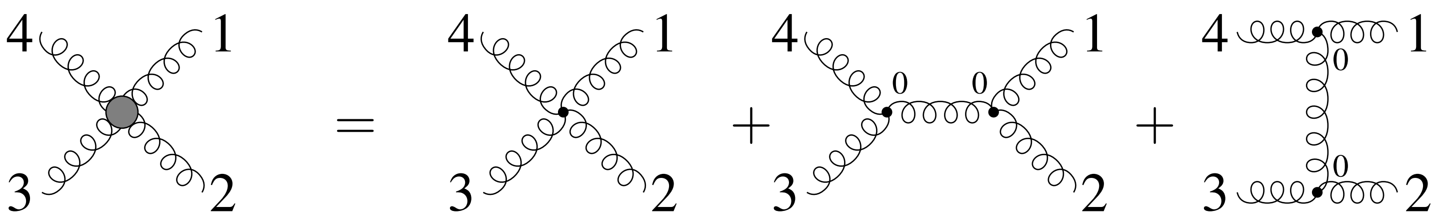
Ch. Schwinn and S.W., JHEP 0505:006, (hep-th/0503015)

Modified vertices

The only non-zero contribution containing ε^0 is obtained from a contraction of a single ε^0 into a three-gluon vertex.

In this case the other two helicities are necessarily ε^+ and ε^-.

The additional polarisation ε^0 can be absorbed into a redefinition of the four-gluon vertex.



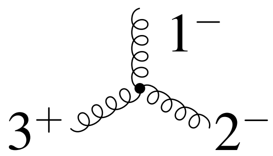
Scalar diagrammatic rules

Extension to massive and massless quarks: Born amplitudes in QCD can be computed from scalar propagators and a set of three- and four-valent vertices. Only vertices of degree zero and one occur.

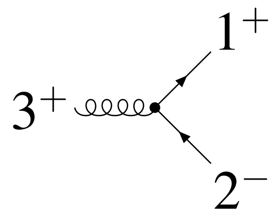
Propagators:

$$\frac{i}{p^2 - m^2}$$

Vertices:



$$= i\sqrt{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$



$$= i\sqrt{2}\frac{[13]^2}{[12]}.$$

Part III : Applications

- Analytical structure of non-MHV amplitudes
- Numerical methods
- Loop amplitudes
- Massive quarks

Analytical structure of non-MHV amplitudes

Degree of an amplitude: number of negative helicity partons minus one.

- On-shell amplitudes of degree zero vanish.
- For amplitudes of degree one: Parke-Taylor formula
- Complexity of the final result increases with the degree: An amplitude of degree two is build from two degree one pieces, etc.

$$A_6^{tree}(1^+,2^+,3^+,4^-,5^-,6^-) =$$
$$4i\left[\frac{\langle 6-|1+2|3-\rangle^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{126}\langle 2-|1+6|5-\rangle}+\frac{\langle 4-|5+6|1-\rangle^3}{\langle 23\rangle\langle 34\rangle[56][61]s_{156}\langle 2-|1+6|5-\rangle}\right]$$

Numerical methods

Compare algorithms based on different methods for the numerical computation of the Born gluon amplitude:

n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00011	0.00043	0.0015	0.005	0.016	0.047	0.13	0.37	1
Scalar	0.00014	0.00083	0.0033	0.011	0.033	0.097	0.26	0.7	1.8
MHV	0.00001	0.00053	0.0056	0.073	0.62	3.67	29	217	—
BCF	0.00002	0.00007	0.0004	0.003	0.017	0.083	0.47	2.5	14.5

CPU time in seconds for the computation of the n gluon amplitude on a standard PC (Pentium IV with 2 GHz), summed over all helicities.

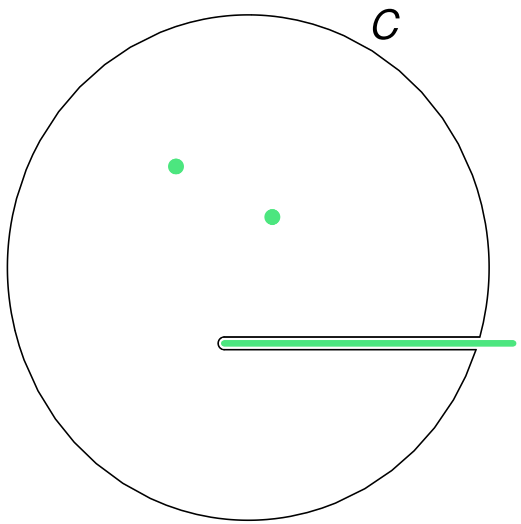
All methods give identical results within an accuracy of 10^{-12}.

M. Dinsdale, M. Ternick and S.W., in preparation

Loop amplitudes

Split QCD amplitudes into $N=4$ and $N=1$ SUSY pieces and a scalar part.

Loop amplitudes have branch cuts:
Get branch cuts from the unitarity method.
Use recursion relations for the rational pieces.



$$A_n(0) \quad = \quad C_\infty - \sum_{poles} \text{res} \frac{A_n(z)}{z} - \int_{B_0}^{\infty} \frac{dz}{z} \text{Disc}\, A_n(z)$$

Complications: Boundary terms, double poles.

Brandhuber, Spence and Travaglini;

Bern, Dixon, Kosower

One-loop corrections $A_n^{1-loop}(1^-,2^-,3^+,\ldots,n^+)$ to adjacent MHV amplitudes have been calculated.

Forde, Kosower

Massive scalars and massive quarks

All-multiplicity Born amplitudes with massive scalars:

$$A_n(\bar{\phi}_1^+, g_2^+, ..., g_{n-1}^+, \phi_n^-), \qquad A_n(\bar{\phi}_1^+, g_2^+, ..., g_{n-1}^-, \phi_n^-).$$

(D. Forde and D.A. Kosower, hep-th/0507292)

Simple relation between amplitudes with massive scalars and massive quarks (top-quarks), based on supersymmetry:

$$A_n(\bar{Q}_1^+, g_2^+, ..., g_{n-1}^+, Q_n^-) \quad = \quad \frac{\langle nq \rangle}{\langle 1q \rangle} A_n(\bar{\phi}_1^+, g_2^+, ..., g_{n-1}^+, \phi_n^-),$$

(Ch. Schwinn and S.W., hep-th/0602012)

Summary

- Standard techniques: Colour decomposition, spinor methods, supersymmetric relations, recurrence relations and the unitarity method

- New developments: Twistor space, MHV vertices, BCF recursion relations and scalar diagrammatic rules

- Applications: analytical, numerical, application to loop amplitudes and to top quark amplitudes