Dimensional Regularization, γ_5 and K-theory

Stefan Weinzierl

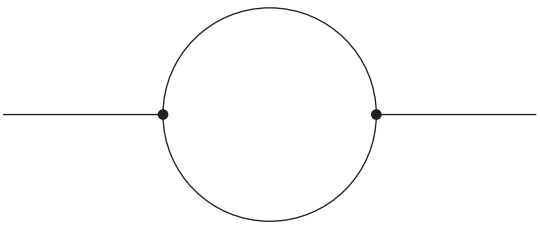
NIKHEF Theory Group

1. **Dimensional regularization**
2. **Constructing a regularization scheme**
   - K-theory
   - Definition of the integration
   - Dirac algebra
3. **Application to loop calculations**
4. **Conclusions**

S.W., "Equivariant dimensional regularization", hep-ph/9903380

Quantum loop corrections

- Loop diagrams are divergent !



$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2} \quad = \quad \pi^2 \int_0^\infty dk^2 \frac{1}{k^2} = \pi^2 \int_0^\infty \frac{dx}{x}$$

This integral diverges at

- $k^2 \to \infty$ (UV-divergence) and at
- $k^2 \to 0$ (IR-divergence).

Regularization schemes

- Cut off:

$$\int_0^\infty dx \to \int_\lambda^\Lambda dx$$

- Pauli-Villars:

$$\frac{1}{k^2 - m^2} \to \frac{1}{k^2 - m^2} - \frac{1}{k^2 - M^2}$$

- Lattice regularization: discretize space-time, this eliminates short distance contributions.

- Dimensional regularization

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^Dk}{(2\pi)^D}$$

Example:

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2)^2} = \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dk^2 \left(k^2\right)^{D/2-3}$$

Dimensional regularization

- Naive dimensional regularization:

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \{\gamma^\mu, \gamma_5\} &= 0 \end{aligned}$$

- The 't Hooft - Veltman scheme:

$$\begin{aligned} k_\mu^{(D)} &= k_\mu^{(4)} + k_\mu^{(-2\varepsilon)} \\ \gamma_\mu^{(D)} &= \gamma_\mu^{(4)} + \gamma_\mu^{(-2\varepsilon)} \end{aligned}$$

γ_5 is defined as a generic four-dimensional object:

$$\gamma_5 = \frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu^{(4)}\gamma_\nu^{(4)}\gamma_\rho^{(4)}\gamma_\sigma^{(4)}$$

Then:

$$\begin{aligned} \left\{\gamma_\mu^{(4)}, \gamma_5\right\} &= 0 \\ \left[\gamma_\mu^{(-2\varepsilon)}, \gamma_5\right] &= 0 \end{aligned}$$

Dimensional reduction

- Dimensional reduction continues the momenta to $D < 4$ dimensions, but keeps spinors and vector fields in four dimensions.

$$\{\gamma_\mu, \gamma_\nu\} \quad = \quad 2g^{(4)}_{\mu\nu}$$

- The D-dimensional metric tensor $g^{(D)}_{\mu\nu}$ acts also as a projection operator:

$$g^{(D)\rho}_{\mu} g^{(4)}_{\rho\nu} \quad = \quad g^{(D)}_{\mu\nu}$$

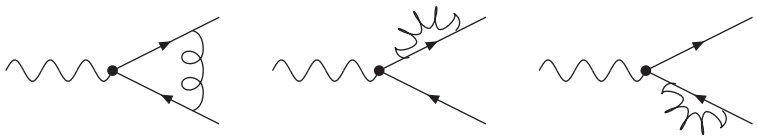
γ_5 in D dimensions

- Naive dimensional regularization is not consistent.

$$(D-4)\text{Tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 \quad = \quad 0$$

- Dimensional reduction is not consistent.

- The 't Hooft-Veltman scheme is consistent.
  - violates the Ward identity for the axial current.



  - this Ward identity has to be restored by hand through a finite renormalization.

Algebraic consistency of dimensional regularization

- Axiomatic approach ('t Hooft, Veltman, Breitenlohner, Maison):
  Take γ_μ, γ_5, $g_{\mu\nu}$ etc. as a set of abstract symbols with a given set of relations. Show that these relations are self-consistent.

- Constructive approach (Wilson, Collins):
  Find a represenation for each object in some (finite or infinit-dimensional) vectorspace.

The constructive approach

Take the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ and consider pairs (p, q) with $p, q \in \mathbb{N}$.

Call two pairs (p, q) and (k, l) equivalent if

$$p + l \quad = \quad k + q$$

Define an addition by components

$$(a, b) + (c, d) \quad = \quad (a + c, b + d)$$

The set of equivalence classes yields the integer numbers.

The inverse of (p, q) is (q, p).

The neutral element is (a, a).

K - theory

- Repeat the exercise with pairs (a, b) and (c, d), where $a, b, c, d \in \mathbb{Z}$, and the equivalence relation

$$a \cdot d \quad = \quad c \cdot b$$

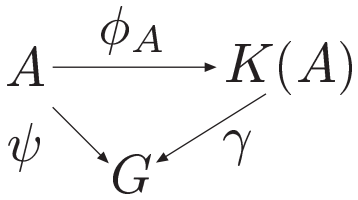
- Consider pairs (V_i, V_j) and (V_k, V_l) from a set of vector spaces $\{V_1, V_2, V_3, ...\}$ and the relation

$$V_i \oplus V_l \quad \sim \quad V_k \oplus V_j$$

This construction is quite general and forms the basis of what mathematicians call K - theory.

The K - functor

Let A be an abelian semi-group. The K-functor associates to A an abelian group $K(A)$ with the property that there is a canonical semi-group homomorphism $\phi_A : A \to K(A)$ such that for any group G and semi-group homomorphism $\psi : A \to G$, there is a unique homomorphism $\gamma : K(A) \to G$ such that $\psi = \gamma\phi_A$.



Construction of $K(A)$: Consider the equivalence relation on the set-theoretical product $A \times A$. We put $(a, b) \sim (a', b')$ when there exists a $p \in A$ such that

$$a + b' + p \quad = \quad a' + b + p.$$

Then by definition $K(A) = A \times A/\sim$.

More contructivism

Let $\mathcal{V} = \{V_1, V_2, ..., V_i, ...\}$ be a set of finite-dimensional vector spaces.

- $\mathcal{V}$ is an abelian semi-group with respect to the direct sum $\oplus$ and the tensor product $\otimes$.
- Use Grothendieck's K-functor twice to construct the corresponding abelian groups.
- Define the rank of an quadrupel $[(V_i, V_j), (V_k, V_l)]$ by

$$\text{rank} \quad = \quad \frac{\dim V_i - \dim V_j}{\dim V_k - \dim V_l}$$

- Note the difference between the "rank" and $\dim V_i + \dim V_j + \dim V_k + \dim V_l$.
- Can construct quadruples such that their ranks form a dense subset of $\mathbb{R}$.
- This construction can be extended to the complex case.
- Can work with vectorspace of even dimension only.

Integration

Choose a quadrupel of vectorspaces of rank r and whose sum of dimension is $2m$. Define the integration by

$$\int d^r k f(k^2) =$$

$$\frac{\pi^{r/2-m}}{\Gamma(r/2-m)} \int d^{2m}k \int_0^\infty dk_\perp^2 \left(k_\perp^2\right)^{r/2-m-1} f(k^2+k_\perp^2)$$

- The integration is independent of the chosen representative.
- Linearity:
$$\int d^r k(af_1 + bf_2) = a\int d^r k f_1 + b\int d^r k f_2$$
- Translation invariance:
$$\int d^r k f(k+q) = \int d^r k f(k)$$
- Scaling law:
$$\int d^r k f(\lambda k) = \lambda^{-r} \int d^r k f(k)$$
- Normalization:
$$\int d^r k \exp(-k^2) = \pi^{r/2}$$

The Dirac algebra

Consider

$$V_4 \oplus V_{-2\varepsilon},$$

where V_4 is a four-dimensional vector space and $V_{-2\varepsilon}$ a quadrupel of rank -2ε and total dimension $2m$. Recall that we can work with vectorspaces of even dimension only.

- Option HV: Define Dirac matrices of dimensions $2^{2+m} \times 2^{2+m}$ over $(V_4 \oplus V_{-2\varepsilon})$. With

  $$\gamma_5 \quad = \quad i\gamma^0\gamma^1\gamma^2\gamma^3$$

  this yields the 't Hooft - Veltman scheme.

- Option FD: Define Dirac matrice of dimensions $2^2 \times 2^2$ over V_4 and of dimensions $2^m \times 2^m$ over $V_{-2\varepsilon}$ separately. This corresponds to a four-dimensional scheme.

The four-dimensional scheme in practice

- The scheme is consistent.
- The Dirac algebra can effectively be performed in four dimensions.
- For the loop momentum one has

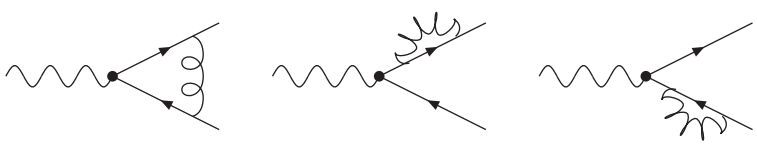
$$\not{k}\not{k} \quad = \quad k^2_{(4-2\varepsilon)} - k^2_{(-2\varepsilon)}$$

- The scheme violates Ward identities, which have to be restored by finite renormalizations.

Finite Renormalization

A regularization scheme may violate Ward identities, which have to be restored by finite renormalizations.

Example: The vector current (V) and the non-singlet axial current (A)

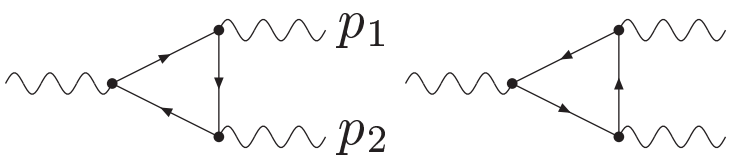


Finite renormalizations to restore the Ward identities in the HV- and in the FD-scheme:

	V	A
HV	1	$1 - 4\frac{\alpha_s C_F}{4\pi}$
FD	$1 + \frac{\alpha_s C_F}{4\pi}$	$1 + \frac{\alpha_s C_F}{4\pi}$

Anomalies

The triangle graphs for the anomaly:



	AVV	VVA	AAA
HV	1	0	$1/3$
FD	$1/3$	$1/3$	$1/3$

Coefficient of $8i\varepsilon_{\alpha\beta\lambda\kappa}p_1^\lambda p_2^\kappa/(4\pi)^2$ for the AVV-, VVA- and AAA-triangle anomalies in the HV and in the FD scheme. Agreement with Bardeen's result.

Tensor integral reduction

- A next-to-leading order calculation requires the reduction of tensor loop integrals to standard scalar integrals.

- Modern approaches rely on spinor methods: Basic objects are massless two-component Weyl spinors of fixed helicity.

- Fierz and Schouten identities yield efficient reduction algorithms, which avoid the appearances of Gram determinates.

- The four-dimensional scheme fits naturally in this framework.

Traditional techniques

- Calculate

$$2 \text{ Re } A^*_{Born} A_{Loop}$$

- Use Pasarino-Veltman reduction for tensor integrals.

- Use 't Hooft-Veltman scheme for γ_5.

Solves the problem in principal, but leads to complicated expressions !

Spinor methods

- Basic objects: Massless two-component Weyl spinors

$$|p\pm\rangle, \qquad \langle p\pm|$$

- Gluon polarization vectors:

$$\varepsilon_\mu^+(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

- Massive spinors:

$$\begin{aligned} u(\pm) &= \frac{1}{\sqrt{2pq}}\,(\not{p}+m)\,|q\mp\rangle \\ \bar{u}(\pm) &= \frac{1}{\sqrt{2pq}}\langle q\mp|\,(\not{p}+m) \end{aligned}$$

Tensor integral reduction

- The Pasarino-Veltman algorithm based on Lorentz invariance.
- The Feynman parameter space technique by Z. Bern, L. Dixon and D.A. Kosower.
- Partial integration (K.G. Chetyrkin and F.V. Tkachov)
- Dual vectors In four dimensions there can be only four linearly independent vectors (J.A.M. Vermaseren and W. van Neerven).
- Spinor algebra Loop momenta are usually sandwiched between spinors (R. Pittau).

Tensor integral reduction with spinors

Example:

$$\gamma^\mu\gamma^\nu \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2-m_1^2)...((k-q_n)^2-m_n^2)}$$

Trick:

$$\gamma^\mu\gamma^\nu k_\mu k_\nu \quad = \quad (k^2-m_1^2)+m_1^2$$

Basic identities:

- Fierz identity:

$$\langle A+|\gamma_\mu|B+\rangle\langle C-|\gamma^\mu|D-\rangle \quad = \quad 2\,[AD]\,\langle CB\rangle$$

- Schouten identity for spinors:

$$\langle AB\rangle\langle CD\rangle \quad = \quad \langle AD\rangle\langle CB\rangle + \langle AC\rangle\langle BD\rangle$$

- Schouten identity for four-vectors:

$$\varepsilon(q_1,q_2,q_3,q_4)k_\mu \quad = \quad \sum_{a=1}^{4}(k\cdot q_a)\,v_\mu^a$$

$$v_\mu^a = \frac{1}{6}\varepsilon^{abcd}\varepsilon(\mu,q_b,q_c,q_d)$$

Tensor integral reduction with spinors

Step 1: Rank ≥ 2 (R. Pittau).

$$\langle p - |\not{k}|q-\rangle\langle p + |\not{k}|q+\rangle \quad = \quad \text{rank-1}$$

Step 2: Rank $= 1$ (S.W.).

Box : $\quad \text{Tr}\, \not{k}\not{p}_1\not{p}_2\not{p}_3 \quad = \quad$ scalar

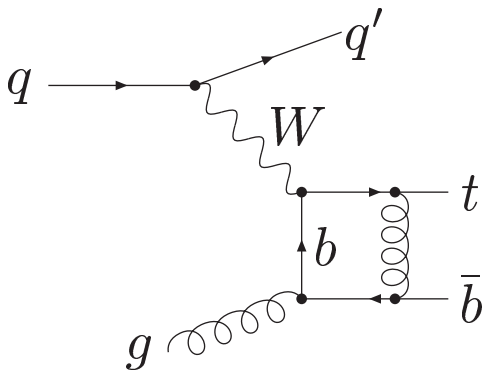
Pentagon : $\quad \text{Tr}\, \not{k}\not{p}_1\not{p}_2\not{p}_3\not{p}_4\not{p}_5 \quad = \quad$ scalar

No Gram determinants involved !

Single top production

Single top production probes the W-t-b vertex, which might reveal new physics.

NLO corrections to W-gluon fusion:



- W-boson has left-handed couplings.
- Top quark is massive.

Work in progress: B.Harris, E.Laenen, L.Phaf, Z.Sullivan and S.W.

Conclusions

- Quantum corrections require a regularization scheme.
- The scheme has to be consistent.
- K-theory can be used for the construction.
- The scheme should be easy to use in practical calculations.