Trading a Calabi-Yau three-fold for a curve of genus two

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work in progress

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Motivation

- **Physics** profits from a fruitful interplay with **geometry**.
- This also applies to perturbative quantum field theory.
- **Feynman integrals** related to **geometric objects**: Spheres, elliptic curves, curves of higher genus, Calabi-Yaus, ...
- We want to learn and understand as much as possible.

Content

- In this talk we will study the simplest Feynman integral related to a Calabi-Yau threefold: The equal-mass four-loop banana integral.
- It is known that the maximal cut of the equal-mass four-loop banana integral is a **period of a Calabi-Yau three-fold**.
- This talk:
  - The maximal cut is also the **period of a genus-two curve**.
  - This curve can be constructed explicitly.
  - The curve varies holomorphically with $z = m^2/p^2$.

More precisely:

- On the Calabi-Yau side:
  - The Calabi-Yau threefold has $h^{2,1} = 1$, hence $\dim H^3(Y) = 4$.
  - We may integrate the holomorphic $(3,0)$-form Ω against four independent cycles. This yields four integral periods.
  - These four integral periods are annihilated by a Picard-Fuchs operator $L^{(0)}$ of degree four.
- On the side of the genus two curve:
  - We construct a holomorphic one-form as a linear combination

$$\omega = c_0 \, \omega_0 + c_1 \, \omega_1 .$$

  - We may integrate the holomorphic $(1,0)$-form ω against four cycles (two a-cycles and two b-cycles).
  - We show that the periods so obtained are again annihilated by the same Picard-Fuchs operator $L^{(0)}$.

Notation

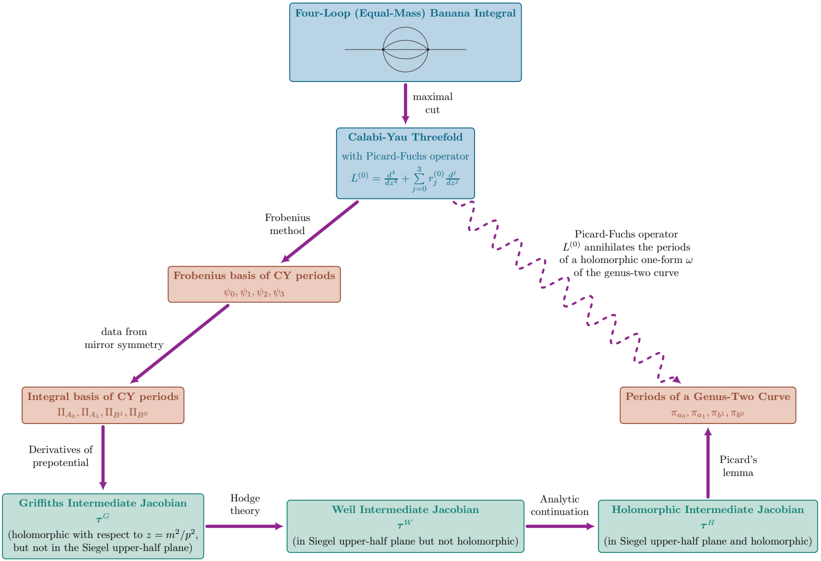
- Capital letters on the Calabi-Yau manifold Y:

Ω:	Holomorphic $(3,0)$-form
A_0, A_1, B^0, B^1:	Symplectic basis of $H_3(Y, \mathbb{Z})$
$\Pi_{A_0}, \Pi_{A_1}, \Pi_{B^0}, \Pi_{B^1}$:	Periods of Ω

- Lower-case letters on the genus-two curve C:

ω:	Holomorphic $(1,0)$-form
a_0, a_1, b^0, b^1:	Symplectic basis of $H_1(C, \mathbb{Z})$
$\pi_{a_0}, \pi_{a_1}, \pi_{b^0}, \pi_{b^1}$:	Periods of ω

Outline



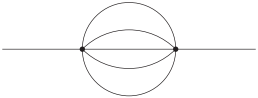
Four-Loop (Equal-Mass) Banana Integral
maximal cut
Calabi-Yau Threefold
with Picard-Fuchs operator
$L^{(0)} = \frac{d^4}{dz^4} + \sum_{j=0}^{3} r_j^{(0)} \frac{d^j}{dz^j}$
Frobenius method
Frobenius basis of CY periods
$\psi_0, \psi_1, \psi_2, \psi_3$
data from mirror symmetry
Integral basis of CY periods
$\Pi_{A_0}, \Pi_{A_1}, \Pi_{B^1}, \Pi_{B^0}$
Derivatives of prepotential
Griffiths Intermediate Jacobian
τ^G
(holomorphic with respect to $z = m^2/p^2$, but not in the Siegel upper-half plane)
Hodge theory
Weil Intermediate Jacobian
τ^W
(in Siegel upper-half plane but not holomorphic)
Analytic continuation
Holomorphic Intermediate Jacobian
τ^H
(in Siegel upper-half plane and holomorphic)
Picard's lemma
Periods of a Genus-Two Curve
$\pi_{a_0}, \pi_{a_1}, \pi_{b^1}, \pi_{b^0}$
Picard-Fuchs operator $L^{(0)}$ annihilates the periods of a holomorphic one-form ω of the genus-two curve

Section 1

Physics

The equal-mass four-loop banana integtral

The object of interest: The family of the **equal-mass four-loop banana integrals**:



These integrals depend on one kinematic variable

$$z = \frac{m^2}{p^2}.$$

There are $1+4=5$ master integrals, a possible basis is given by

$$I_{11110}, \; I_{11111}, \; I_{11112}, \; I_{11113}, \; I_{11114}.$$

Bönisch, Duhr, Fischbach, Forum, Görges, Klemm, Kreimer, Nega, Pögel, Safari, Tancredi, von Hippel, Wagner, Wang, S.W., '19 - '23

The ε-factorised form

- There is a basis J, which puts the **differential equation** for this family into an **ε-factorised form**

$$\frac{d}{dz} J = \varepsilon A(z) J.$$

Pögel, Wang, S.W. '22

- In order to construct this basis one important ingredient is to study the maximal cut of I_{11111} in two space-time dimensions.
- The maximal cut of I_{11111} satisfies a fourth-order differential equation.

The Picard-Fuchs operator

- The **Picard-Fuchs operator**:

$$
\begin{aligned}
L^{(0)} \;=\; & \frac{d^4}{dz^4} + \left[\frac{2}{z} - 2\frac{1}{(1-z)} - 2\frac{9}{(1-9z)} - 2\frac{25}{(1-25z)} \right] \frac{d^3}{dz^3} \\
& + \frac{1-98z+1839z^2-3150z^3}{z^2(1-z)(1-9z)(1-25z)} \frac{d^2}{dz^2} - \frac{\left(1+15z-60z^2\right)(1-15z)}{z^3(1-z)(1-9z)(1-25z)} \frac{d}{dz} \\
& + \frac{1-5z}{z^4(1-z)(1-9z)(1-25z)}.
\end{aligned}
$$

- We are interested in the differential equation

$$L^{(0)} \psi \;=\; 0$$

  and the **interpretation of the solutions as periods of geometric objects**.

- $L^{(0)}$ is related to operator 34 in the list of Almkvist, van Enckevort, van Straten and Zudilin (and hence a Calabi-Yau operator).

The Frobenius solution

The point $z = 0$ is a **point of maximal unipotent monodromy**. From the method of Frobenius it follows that we may write the 4 independent solutions as

$$\begin{aligned}
\psi_0 &= \sum_{n=0}^{\infty} a_{0,n} z^{n+1}, \\
\psi_1 &= \frac{1}{(2\pi i)} \sum_{n=0}^{\infty} \left[a_{1,n} + a_{0,n} \ln z \right] z^{n+1}, \\
\psi_2 &= \frac{1}{(2\pi i)^2} \sum_{n=0}^{\infty} \left[a_{2,n} + a_{1,n} \ln z + \frac{1}{2} a_{0,n} \ln^2 z \right] z^{n+1}, \\
\psi_3 &= \frac{1}{(2\pi i)^3} \sum_{n=0}^{\infty} \left[a_{3,n} + a_{2,n} \ln z + \frac{1}{2} a_{1,n} \ln^2 z + \frac{1}{6} a_{0,n} \ln^3 z \right] z^{n+1}.
\end{aligned}$$

Section 2

The Calabi-Yau story

The Calabi-Yau threefold

- Vanishing of the second graph polynomial $\mathcal{F}$ in $\mathbb{CP}^4$:

$$Y^{\mathrm{sing}} = \left\{ [a_1 : a_2 : a_3 : a_4 : a_5] \in \mathbb{CP}^4 \mid \mathcal{F}(a) = 0 \right\}.$$

This defines a **singular Calabi-Yau threefold**.

- **Hulek-Verrill variety**: There is a smooth projective Calabi-Yau threefold Y, birational to Y^{sing}, defined as the toric compactification of the locus

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = z \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} \right) + \frac{1}{a_6} = 0$$

on $\mathbb{T}^5 = \mathbb{CP}^5 \setminus \{ a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 = 0 \}$.

Hulek, Verrill, '05

The Hodge diamonds

			1			
		0		0		
	0		9		0	
1		1		1		1
	0		9		0	
		0		0		
			1			

Calabi-Yau manifold Y

			1			
		0		0		
	0		1		0	
1		9		9		1
	0		1		0	
		0		0		
			1			

Mirror manifold Y^{mirror}

Data from the Frobenius solutions

- The **mirror map**:

$$\tau = \frac{\psi_1}{\psi_0}, \qquad q = e^{2\pi i \tau}.$$

Candelas, De La Ossa, Green, Parkes '91

- The differential operator $L^{(0)}$ is a Calabi-Yau operator and has one non-trivial ***Y*-invariant**:

$$Y_2 \quad = \quad \frac{d^2}{d\tau^2} \frac{\psi_2}{\psi_0}.$$

We may write Y_2 in the form

$$Y_2 \quad = \quad \frac{1}{24} \left(q \frac{d}{dq} \right)^3 \left[4 \ln^3 q + \sum_{k=1}^{\infty} n_k \mathrm{Li}_3\left(q^k\right) \right].$$

The n_k are integer numbers.

M. Bogner '13, D. van Straten '17

The special local normal form

- The differential operator $L^{(0)}$ can be written in the q-coordinate as

$$L^{(0)} = \beta \left(q \frac{d}{dq} \right)^2 \frac{1}{Y_2} \left(q \frac{d}{dq} \right)^2 \frac{1}{\psi_0}$$

where β is a function of q.

- The operator

$$N\left(L^{(0)}\right) = \left(q \frac{d}{dq} \right)^2 \frac{1}{Y_2} \left(q \frac{d}{dq} \right)^2$$

is called the **special local normal form** of the operator $L^{(0)}$.

M. Bogner, '13

Topological data from the mirror manifold

Triple intersection number κ on Y^{mirror}:

$$\kappa = \int\limits_{Y^{\mathrm{mirror}}} \omega^{\mathrm{Kähler}} \wedge \omega^{\mathrm{Kähler}} \wedge \omega^{\mathrm{Kähler}} = 24.$$

Integrated second Chern class of Y^{mirror}:

$$C_2 = \int\limits_{Y^{\mathrm{mirror}}} c_2 \wedge \omega^{\mathrm{Kähler}} = 24.$$

Euler characteristic χ of Y^{mirror}:

$$\chi = \sum_{p,q} (-1)^{p+q} h^{p,q}\left(Y^{\mathrm{mirror}}\right) = -16.$$

Candelas, de la Ossa, Kuusela, McGovern, '21

With κ, C_2 and χ at hand, we get the **integral periods** from the Frobenius solutions:

$$\begin{pmatrix} \Pi_{A_0} \\ \Pi_{A_1} \\ \Pi_{B^1} \\ \Pi_{B^0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{C_2}{24} & 0 & -\kappa & 0 \\ \frac{\chi\zeta_3}{(2\pi i)^3} & \frac{C_2}{24} & 0 & \kappa \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

Normalised integral periods:

$$\hat{\Pi}_J = \frac{\Pi_J}{\Pi_{A_0}}, \qquad J \in \{A_0, A_1, B^1, B^0\}.$$

Special Kähler geometry

There is a **prepotential** $F(\tau)$ and a projective version $F^{\mathrm{proj}}(X^0, X^1)$

$$F(\tau) = F^{\mathrm{proj}}(1,\tau), \qquad F^{\mathrm{proj}}\left(X^0, X^1\right) = \left(X^0\right)^2 F\left(\frac{X^1}{X^0}\right)$$

such that

$$\hat{\Pi}_{B^i} = \left.\frac{\partial F^{\mathrm{proj}}}{\partial X^i}\right|_{(X^0,X^1)=(1,\tau)}.$$

The prepotential works out to

$$F = -4\tau^3 + \tau - 8\frac{\zeta_3}{(2\pi i)^3} - \frac{1}{(2\pi i)^3}\sum_{k=1}^{\infty} n_k \,\mathrm{Li}_3\left(q^k\right).$$

Section 3

Intermediate Jacobians

Intermediate Jacobians

- We will now be considering **intermediate Jacobians** of (complex) dimension 2 for the Calabi-Yau threefold Y. They are all given by

$$J_2 = \mathbb{C}^2 / \left(\mathbb{Z}^2 + \boldsymbol{\tau} \mathbb{Z}^2 \right),$$

  where $\boldsymbol{\tau}$ is a symmetric 2×2-matrix (which should not be confused with τ).
- The **Jacobian variety** of a curve of genus 2 is of a similar form. In this case $\boldsymbol{\tau} \in \mathbb{H}^2$ (Siegel upper half-plane).
- We will interpret one particular intermediate Jacobian as the Jacobian variety of curve of genus 2.

Griffiths' intermediate Jacobian

First try: **Griffiths' intermediate Jacobian**, can be obtained from the prepotential:

$$J_2^G \quad = \quad \mathbb{C}^2 / \left(\mathbb{Z}^2 + \boldsymbol{\tau}^G \mathbb{Z}^2 \right),$$

where

$$\boldsymbol{\tau}^G = \begin{pmatrix} \tau_{00}^G & \tau_{01}^G \\ \tau_{01}^G & \tau_{11}^G \end{pmatrix}, \qquad \tau_{ij}^G = \left. \frac{\partial^2 F^{\mathrm{proj}}}{\partial X^i \partial X^j} \right|_{(X^0, X^1) = (1, \tau)}.$$

But: $\mathrm{Im}(\boldsymbol{\tau}^G)$ **not positive definite**, hence $\boldsymbol{\tau}^G \notin \mathbb{H}^2$, cannot be interpreted as the Jacobian variety of a genus 2 curve.

Complex structures on $H^3(X, \mathbb{R})$

- Alternativ definition of Griffiths' intermediate Jacobian: Start from the cohomology group $H^3(X, \mathbb{R})$ with real coefficients, put a complex structure on it and mod out $H^3(X, \mathbb{Z})$.
- There are **two possibilities** in defining a complex structure on $H^3(X, \mathbb{R})$. One possibility gives us Griffiths intermediate Jacobian defined before, the other possibility gives Weil's intermediate Jacobian.

	$H^{(3,0)}$	$H^{(2,1)}$	$H^{(1,2)}$	$H^{(0,3)}$
Eigenvalue C^G	$+i$	$+i$	$-i$	$-i$
Eigenvalue C^W	$-i$	$+i$	$-i$	$+i$

Weil's intermediate Jacobian

Weil's intermediate Jacobian is given by

$$J_2^W \quad = \quad \mathbb{C}^2 / \left(\mathbb{Z}^2 + \boldsymbol{\tau}^W \mathbb{Z}^2\right),$$

where

$$\boldsymbol{\tau}^W = \begin{pmatrix} \tau_{00}^W & \tau_{01}^W \\ \tau_{01}^W & \tau_{11}^W \end{pmatrix}, \qquad \tau_{ij}^W = -\overline{F_{ij}} - 2i \left. \frac{I_{ik} X^k I_{jl} X^l}{X^m I_{mn} X^n} \right|_{(X^0,X^1)=(1,\tau)}$$

and

$$F_{ij} = \left. \frac{\partial^2 F^{\mathrm{proj}}}{\partial X^i \partial X^j} \right|_{(X^0,X^1)=(1,\tau)}, \qquad I_{ij} = \mathrm{Im}\, F_{ij}.$$

Weil's intermediate Jacobian

- We now have $\boldsymbol{\tau}^W \in \mathbb{H}^2$, hence we may interpret J_2^W as the Jacobian variety of a genus 2 curve.
- But J_2^W **varies non-holomorphically with** z.
- However, non-holomorphic terms drop out in

$$\psi_0 \begin{pmatrix} 1 & \tau \end{pmatrix} \begin{pmatrix} 1 & 0 & \tau_{00}^W & \tau_{01}^W \\ 0 & 1 & \tau_{01}^W & \tau_{11}^W \end{pmatrix} = \begin{pmatrix} \Pi_{A_0} & \Pi_{A_1} & -\Pi_{B^0} & -\Pi_{B^1} \end{pmatrix}.$$

Observation

Let us now restrict to $z \in]0, z_{\mathrm{max}}[$.

Restricted to this **line segment** we have

$$\tau_{00}^{W} = -\frac{\left(F - \tau\partial_\tau F\right)\left(2F - 2\tau\partial_\tau F + \tau^2\partial_\tau^2 F\right)}{F - \tau\partial_\tau F + \tau^2\partial_\tau^2 F},$$

$$\tau_{01}^{W} = -\frac{F\partial_\tau F - \tau\left(\partial_\tau F\right)^2 + \tau F\partial_\tau^2 F}{F - \tau\partial_\tau F + \tau^2\partial_\tau^2 F},$$

$$\tau_{11}^{W} = \frac{\left(F - \tau\partial_\tau F\right)\partial_\tau^2 F}{F - \tau\partial_\tau F + \tau^2\partial_\tau^2 F}.$$

The holomorphic Jacobian

Let us now consider a complex neighbourhood of the line segment $]0, z_{\mathrm{max}}[$. In this neighbourhood we define

$$J_2^H \;=\; \mathbb{C}^2 / \left(\mathbb{Z}^2 + \boldsymbol{\tau}^H \mathbb{Z}^2 \right)$$

through

$$\tau_{00}^H = -\frac{(F - \tau\partial_\tau F)\left(2F - 2\tau\partial_\tau F + \tau^2 \partial_\tau^2 F\right)}{F - \tau\partial_\tau F + \tau^2\partial_\tau^2 F}, \quad \tau_{01}^H = -\frac{F\partial_\tau F - \tau\left(\partial_\tau F\right)^2 + \tau F \partial_\tau^2 F}{F - \tau\partial_\tau F + \tau^2\partial_\tau^2 F}, \quad \tau_{11}^H = \frac{(F - \tau\partial_\tau F)\,\partial_\tau^2 F}{F - \tau\partial_\tau F + \tau^2\partial_\tau^2 F}.$$

Properties:

- J_2^H varies holomorphically with z.
- $\boldsymbol{\tau}^H \in \mathbb{H}^2$.
- One linear combination is annihilated by the Picard-Fuchs operator:

$$\psi_0 \begin{pmatrix} 1 & \tau \end{pmatrix} \begin{pmatrix} 1 & 0 & \tau_{00}^H & \tau_{01}^H \\ 0 & 1 & \tau_{01}^H & \tau_{11}^H \end{pmatrix} = \begin{pmatrix} \Pi_{A_0} & \Pi_{A_1} & -\Pi_{B^0} & -\Pi_{B^1} \end{pmatrix}.$$

Section 4

The genus two curve

Construction of the curve: Outline

- We now construct a genus two curve from its Jacobian variety.
- We take the Jacobian variety to be defined by

$$\boldsymbol{\tau} \;=\; \boldsymbol{\tau}^{H}$$

- The construction of the genus two curve from its Jacobian variety can be done with the help of a lemma from Picard.
- This lemma uses theta functions.

Theta functions

For $\boldsymbol{\tau} \in \mathbb{H}^g$ and $z \in \mathbb{C}^g$ the theta function is defined by

$$\vartheta(z, \boldsymbol{\tau}) = \sum_{n \in \mathbb{Z}^g} e^{i\pi\left(n^T \boldsymbol{\tau} n + 2n^T z\right)}$$

Theta functions with characteristic are defined for $a, b \in \mathbb{Q}^g$ by

$$\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](z, \boldsymbol{\tau}) = \sum_{n \in \mathbb{Z}^g} e^{i\pi\left((n+a)^T \boldsymbol{\tau} (n+a) + 2(n+a)^T (z+b)\right)}.$$

- Of particular importance is the case, where $a, b \in (\mathbb{Z}/2)^g$.
- In this case $4a^T \cdot b$ is an integer, and the characteristic is called even (respectively odd) if this integer is even (respectively odd).

Theta constants

- Let us now specialise to $g = 2$.
- In this case we have 10 even characteristics and 6 odd characteristics.
- Short-hand notation:

$$\theta_1 = \vartheta\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(0,\boldsymbol{\tau}), \qquad \theta_2 = \vartheta\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}(0,\boldsymbol{\tau}),$$

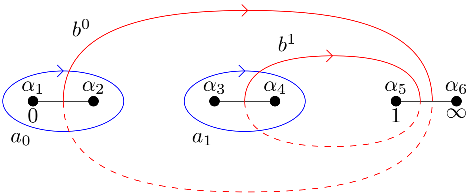
$$\theta_3 = \vartheta\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}(0,\boldsymbol{\tau}), \qquad \theta_4 = \vartheta\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}(0,\boldsymbol{\tau}),$$

$$\theta_5 = \vartheta\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}(0,\boldsymbol{\tau}), \qquad \theta_6 = \vartheta\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}(0,\boldsymbol{\tau}),$$

$$\theta_7 = \vartheta\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}(0,\boldsymbol{\tau}), \qquad \theta_8 = \vartheta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}(0,\boldsymbol{\tau}).$$

$$\theta_9 = \vartheta\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}(0,\boldsymbol{\tau}), \qquad \theta_{10} = \vartheta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}(0,\boldsymbol{\tau}).$$

The curve of genus two



Rosenhain form of a genus two curve:

$$C \ : \ v^2 = P_5(u), \quad P_5(u) = u(u-\alpha_2)(u-\alpha_3)(u-\alpha_4)(u-1)$$

Given $\boldsymbol{\tau} \in \mathbb{H}_2$ the branch points $\alpha_2, \alpha_3, \alpha_4$ are given by a lemma from Picard in terms of theta constants

$$\alpha_2 = \frac{\theta_5^2\theta_6^2}{\theta_1^2\theta_4^2}, \qquad \alpha_3 = \frac{\theta_6^2\theta_7^2}{\theta_4^2\theta_8^2}, \qquad \alpha_4 = \frac{\theta_5^2\theta_7^2}{\theta_1^2\theta_8^2}.$$

The one-form ω

On a genus two curve we have two holomorphic one-forms

$$\omega_0 = \frac{du}{\sqrt{P_5(u)}}, \qquad \omega_1 = \frac{u\,du}{\sqrt{P_5(u)}}.$$

Consider

$$\omega = c_0 \omega_0 + c_1 \omega_1$$

with

$$c_0 = \frac{\theta_2\theta_3\theta_5\theta_6\theta_7\theta_9\theta_{10}}{2\theta_1^2\theta_4^2\theta_8^2} \frac{\psi_0 \partial_1 \theta_{11} - \psi_1 \partial_0 \theta_{11}}{\partial_0\theta_{16}\partial_1\theta_{11} - \partial_0\theta_{11}\partial_1\theta_{16}}, \qquad c_1 = \frac{\theta_2\theta_3\theta_9\theta_{10}}{2\theta_1\theta_4\theta_8} \frac{\psi_0 \partial_1 \theta_{16} - \psi_1 \partial_0 \theta_{16}}{\partial_0\theta_{16}\partial_1\theta_{11} - \partial_0\theta_{11}\partial_1\theta_{16}}.$$

and the odd theta constants

$$\partial_i\theta_{11} = \frac{\partial}{\partial z_i} \vartheta \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} (z, \boldsymbol{\tau}) \Big|_{z=0}, \qquad \partial_i\theta_{16} = \frac{\partial}{\partial z_i} \vartheta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} (z, \boldsymbol{\tau}) \Big|_{z=0}.$$

The periods of ω are annihilated by the Picard-Fuchs operator $L^{(0)}$.

Section 5

Wrap-up

Calculational steps

- From the differential equation get the Frobenius solution.
- From the Frobenius solution get τ and the prepotential F.
- From τ and F get $\boldsymbol{\tau}^H \in \mathbb{H}^2$.
- From $\boldsymbol{\tau}^H$ get the branchpoints α_2, α_3, α_4.

$$C \quad : \quad v^2 \; = \; u(u-\alpha_2)(u-\alpha_3)(u-\alpha_4)(u-1),$$

Hierarchy of small parameters

For small z we have approximately

$$e^{i\pi\tau_{00}} \approx \exp\left(\frac{\ln^3 z}{2\pi^2}\right), \qquad e^{i\pi\tau_{11}} \approx z^6, \qquad \frac{i}{2}\left(1+e^{i\pi\tau_{01}}\right) \approx -\frac{9\zeta_3}{4\pi\ln z}.$$

All three expressions on the right-hand sides go to zero as $z \to 0$, albeit at different rates.

z	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
$e^{\frac{\ln^3 z}{2\pi^2}}$	$7.4 \cdot 10^{-93}$	$9.6 \cdot 10^{-59}$	$2.7 \cdot 10^{-34}$	$6.5 \cdot 10^{-18}$	$5.6 \cdot 10^{-8}$	$7 \cdot 10^{-3}$
z^6	10^{-42}	10^{-36}	10^{-30}	10^{-24}	10^{-18}	10^{-12}
$-\frac{9\zeta_3}{4\pi\ln z}$	0.053	0.062	0.075	0.093	0.12	0.19

For sufficient small values of z we have the **hierarchy**

$$e^{\frac{\ln^3 z}{2\pi^2}} \ll z^6 \ll -\frac{9\zeta_3}{4\pi\ln z}.$$

Conclusions

- Maximal cut of the equal-mass four-loop banana integral:
  - period of a Calabi-Yau threefold
  - period of a curve of genus two.
- There is a linear combination of holomorphic one-forms, whose periods are annihilated by the Picard-Fuchs operator.
- The curve varies holomorphically.
- Jacobian varieties are useful.
- Outlook: Extensions to odd-dimensional Calabi-Yau manifolds.