Subtraction method for NLO calculations in top quark physics

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1. **NLO calculations: Infrared divergencies**
2. **Phase space slicing and the subtraction method**
3. **Technique for the integration**
4. **Results**
5. **Outlook**

NLO calculations

- LO calculations: one parton = one jet

- NLO calculations receive contributions from

  - one loop virtual corrections
  - real emission with one unresolved parton

  Both of these two contributions are infrared divergent. Only the sum is finite.

In perturbation theory we can only calculate infrared-safe quantities.

General methods

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

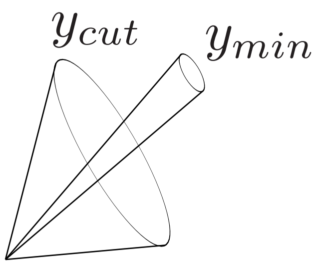
- Phase space slicing
  - e^+e^-: W. Giele and N. Glover, (1992)
  - initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
  - massive partons, fragmentation: S. Keller and E. Laenen, (1999)

- Subtraction method
  - residue approach: S. Frixione, Z. Kunzst and A. Signer, (1995)
  - dipole formalism: S. Catani and M. Seymour, (1996)

The dipole formalism can deal with final- and initial-state singularities and identified hadrons in the final state.
But: No extension to massive partons yet !

Phase space slicing

Splits the integration of the real emission contribution into a region $y > y_{min}$ and a region $y < y_{min}$.



The former is free of singularities and the integration can be performed numerically there.
In the latter the matrix element is approximated and the integration over the one-parton phase space is performed analytically.

- Introduces an error of order y_{min}.
- The first region gives a contribution of the form

$$a \ln^2 y_{min} + b \ln y_{min} + c$$

  The logarithms $\ln^2 y_{min}$ and $\ln y_{min}$ cancel against the contribution from the second region.
- But: Cancelation happens only numerically!

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V$$

$$= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

Colour disentanglement

Full QCD amplitudes do not factorize in the soft limit, but colour-ordered partial amplitudes do.
The $(n+1)$-parton matrix element is approximated by a sum of dipole terms

$$\sum_{pairs\ i,j} \sum_{k \neq i,j} \mathcal{D}_{ij,k} = \sum_{pairs\ i,j} \sum_{k \neq i,j} -\frac{1}{2p_i \cdot p_j}$$

$$\cdot \langle 1, ..., (\tilde{ij}), ..., \tilde{k}, ... | \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k} | 1, ..., (\tilde{ij}), ..., \tilde{k}, ... \rangle$$

$V_{ij,k}$ is a matrix in the helicity space of the emitter with the correct soft and collinear behaviour.
The colour charge operators $\mathbf{T}_i$ for a quark, gluon and antiquark in the final state are

$$\begin{aligned} \text{quark}: &\quad \langle ... q_i ... | T^a_{ij} | ... q_j ... \rangle \\ \text{gluon}: &\quad \langle ... g^c ... | i f^{cab} | ... g^b ... \rangle \\ \text{antiquark}: &\quad \langle ... \bar{q}_i ... | \left(-T^a_{ji} \right) | ... \bar{q}_j ... \rangle \end{aligned}$$

Soft and collinear limits

In the soft limit the $m+1$-parton matrix element behaves according to

$$\begin{aligned}&\langle 1,...,m+1|1,...,m+1\rangle \\ &\quad = \quad -4\pi\mu^{2\varepsilon}\alpha_s\langle 1,...,m|\mathbf{J}^{\mu\dagger}\mathbf{J}_\mu|1,...,m\rangle\end{aligned}$$

$$\begin{aligned}&\mathbf{J}^{\mu\dagger}\mathbf{J}_\mu|1,...,m+1\rangle \\ &= \quad \left(2\sum_{i\neq k}\frac{2p_ip_k}{(2p_ip_j)(2p_jp_k)}\mathbf{T}_i\cdot\mathbf{T}_k+\sum_i\frac{4m_i^2}{(2p_ip_j)^2}\mathbf{T}_i\cdot\mathbf{T}_i\right) \\ &\qquad |1,...,m+1\rangle\end{aligned}$$

If the particles are massless, the matrix element behaves in the collinear limit as

$$\begin{aligned}&\langle 1,...,m+1|1,...,m+1\rangle \\ &\quad = \quad 4\pi\mu^{2\varepsilon}\alpha_s\langle 1,...,m|\frac{2}{2p_ip_j}\hat{P}_{(ij),i}(\hat{z},k_\perp,\varepsilon)|1,...,m\rangle\end{aligned}$$

Extension to massive partons

- Find suitable subtraction terms

- Integrate the subtraction terms over the dipole phase space

- Find a mapping for the momenta from the $(n+1)$-parton configuration to the n-parton configuration

Technique for the integration

Three basic integrals:

- Euler Beta function:

$$\int_0^1 dx x^{a-1}(1-x)^{c-a-1} = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}$$

- Hypergeometric function:

$$\int_0^1 dx x^{a-1}(1-x)^{c-a-1}(1-x_0 x)^{-b}$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} {}_2F_1(a,b;c,x_0)$$

- First Appell function:

$$\int_0^1 dx x^{a-1}(1-x)^{c-a-1}(1-x_1 x)^{-b_1}(1-x_2 x)^{-b_2}$$
$$= \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_1(a,b_1,b_2;c;x_1,x_2)$$

Taylor expansion and iteration

Taylor expansion of the hypergeometric function and of the Appell function allows us to perform the next integral, which is of the same type as above.

$$
{}_2F_1(a, b; c, x_0) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x_0^n}{n!}
$$

$$
\begin{aligned}
&F_1(a, b_1, b_2; c; x_1, x_2) \\
&\quad = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2}}{(c)_{m_1+m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!}
\end{aligned}
$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol.

Expansion of the Gamma function

After all integrals have been performed, we end up with multiple sums involving Gamma-functions depending on $\varepsilon = (4-D)/2$. These Gamma-functions are then expanded according to

$$\frac{\Gamma(n+\varepsilon)}{\Gamma(n)\Gamma(\varepsilon)} = \varepsilon\left(1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)\right.$$
$$\left.+\varepsilon^3 Z_{111}(n-1)+\ldots+\varepsilon^{n-1} Z_{11\ldots1}(n-1)\right)$$

where $Z_{11\ldots1}(n)$ are Euler-Zagier sums defined by

$$Z_{m_1,\ldots,m_k}(n) \quad = \quad \sum_{i_1>i_2>\ldots>i_k>0}^{n} \frac{1}{i_1^{m_1}}\frac{1}{i_2^{m_2}}\cdots\frac{1}{i_k^{m_k}}$$

The structure of the final results

Rearranging summation indices we recognize that all sums fall in the class of Goncharov's multiple polylogarithms

$$\begin{aligned}
&\mathrm{Li}_{m_k,\ldots,m_2,m_1}(x_k,\ldots,x_2,x_1) \\
&= \sum_{i_1>i_2>\ldots>i_k>0}^{\infty} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}
\end{aligned}$$

In the case with only one massive parton it is actually sufficient to restrict oneself to harmonic polylogarithms (Remiddi, Vermaseren)

$$H_{m_1,m_2\ldots,m_k}(x) = \sum_{i_1>i_2>\ldots>i_k>0}^{\infty} \frac{x^{i_1}}{i_1^{m_1}} \frac{1}{i_2^{m_2}} \cdots \frac{1}{i_k^{m_k}}$$

The dipole phase space measure

The dipole phase space measure can be obtained from the three-particle phase space:

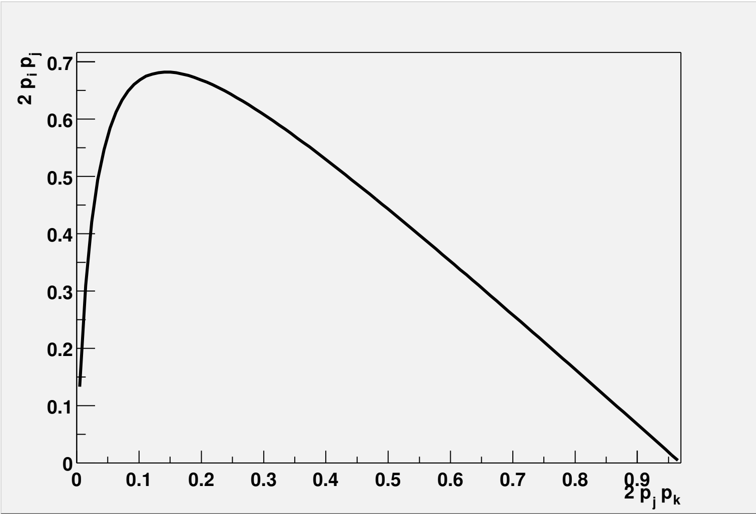
$$\int d\phi_3(P \to p_i, p_j, p_k)$$
$$= \int d\phi_2(P \to \tilde{p}_{ij}, \tilde{p}_k) d\phi_{dipole}(2p_ip_j, 2p_jp_k, 2p_ip_k, \Omega^{(i)}_{D-2})$$

where

$$d\phi_{dipole}(2p_ip_j, 2p_jp_k, 2p_ip_k)$$
$$= (2\pi)^{1-D} \frac{2\pi^{D/2-1}}{\Gamma\left(\frac{D}{2}-1\right)} \frac{1}{4} \left(\left(P^2 - m_i^2 - m_k^2\right)^2 - 4m_i^2 m_k^2 \right)^{\frac{3-D}{2}}$$
$$\int d(2p_ip_j) d(2p_jp_k) d(2p_ip_k)$$
$$\delta\left(P^2 - m_i^2 - m_j^2 - m_k^2 - 2p_ip_j - 2p_jp_k - 2p_ip_k\right)$$
$$\left(-P^2 \lambda\left(|\vec{p}_i|^2, |\vec{p}_j|^2, |\vec{p}_k|^2\right)\right)^{\frac{D-4}{2}} \theta\left(-\lambda\left(|\vec{p}_i|^2, |\vec{p}_j|^2, |\vec{p}_k|^2\right)\right)$$

One massive parton

We need a suitable parameterization of the dipole phase space!



The Dalitz plot for two massless particles ($m_i = m_j = 0$) and one massive particle ($m_k \neq 0$).

The collinear singularity occurs for $2p_ip_j \to 0$, the soft singularity for $2p_ip_j \to 0$ and $2p_jp_k \to 0$.

One massive parton

With $D = 4 - 2\varepsilon$ the dipole phase space becomes

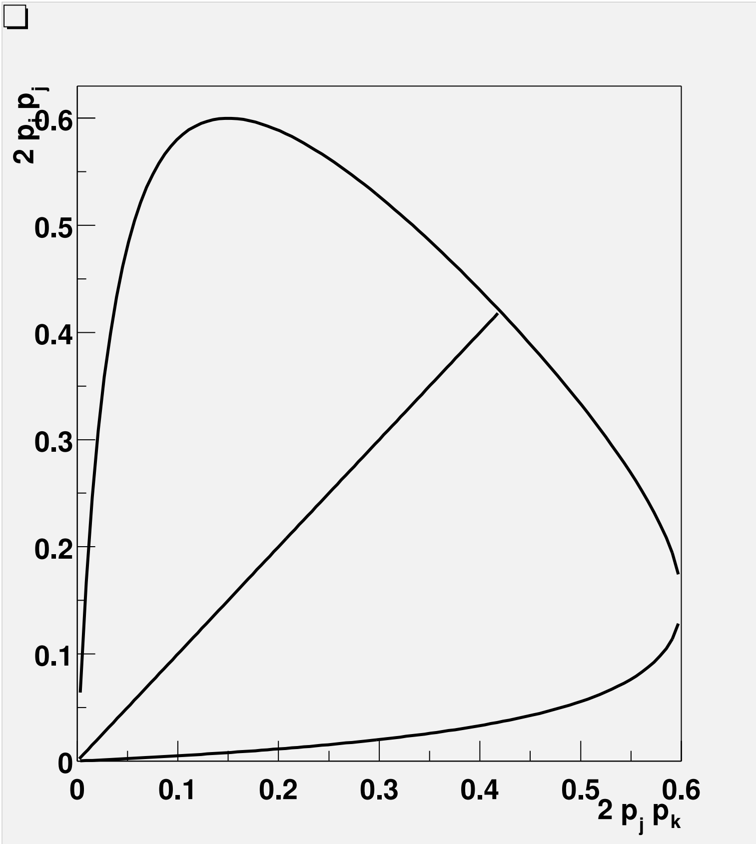
$$d\phi_{dipole} = \frac{(4\pi)^{\varepsilon-2}}{\Gamma(1-\varepsilon)} (u_0)^{2-2\varepsilon} \left(P^2\right)^{1-\varepsilon}$$

$$\int\limits_0^1 du u^{1-2\varepsilon}(1-u)^{1-2\varepsilon} (1-u_0 u)^{-1+\varepsilon} \int\limits_0^1 dv v^{-\varepsilon}(1-v)^{-\varepsilon}$$

where

$$\begin{aligned} u_0 &= \frac{P^2 - m_k^2}{P^2} \\ u &= \frac{2p_i p_j + 2p_j p_k}{2p_i p_k + 2p_i p_j + 2p_j p_k} \\ v &= \frac{2p_i p_j \left(2p_i p_k + m_k^2\right)}{2p_i p_k \left(2p_i p_j + 2p_j p_k\right)} \end{aligned}$$

Two massive partons



The Dalitz plot for one massless particle ($m_j = 0$) and two massive particles ($m_i = m_k = m$).

The soft singularity occurs for $2p_ip_j \to 0$ and $2p_jp_k \to 0$.

The straight line indicates the separation of the region $s > 0$ from the region $s < 0$.

Two massive partons

In the case where one mass vanishes ($m_j = 0$) and the other two are equal ($m_i = m_k = m$) we obtain

$$d\phi_{dipole} = \frac{(4\pi)^{\varepsilon-2}}{\Gamma(1-\varepsilon)} \left(P^2\right)^{1-\varepsilon} (r_0)^{2-2\varepsilon} \, 2^{2\varepsilon-1}$$

$$\int\limits_0^1 dr r^{1-2\varepsilon} (1-r)^{-\varepsilon+1/2} \left(1-r_0 r\right)^{-1/2} \int\limits_{-1}^1 ds \left(1-s^2\right)^{-\varepsilon}$$

where

$$\begin{aligned} r_0 &= 1 - 4\frac{m^2}{P^2} \\ 2p_i p_j &= \frac{1}{2} P^2 r_0 r \left(1 - s\sqrt{\frac{r_0(1-r)}{1-r_0 r}}\right) \\ 2p_j p_k &= \frac{1}{2} P^2 r_0 r \left(1 + s\sqrt{\frac{r_0(1-r)}{1-r_0 r}}\right) \end{aligned}$$

Subtraction terms

- Final-state singularities with no initial-state partons
  - massless emitter, massive spectator
  - massive emitter, massless spectator
  - massive emitter, massive spectator (of equal mass)
- Initial-state partons
  - massless initial emitter, massive final spectator
  - massive final emitter, massless initial spectator

Subtraction terms

Example of a subtraction term corresponding to the splitting $q \to qg$ with a massive spectator:

$$\langle s|V_{q_i g_j,k}|s'\rangle = 8\pi\mu^{2\varepsilon}\alpha_s C_F \delta_{ss'} \left[\frac{2z(1-y)}{1-z(1-y)} + (1-\rho\varepsilon)(1-z(1-y))\right]$$

Integration over the dipole phase space yields:

$$\mathcal{V}_{qg}(u_0,\varepsilon) = C_F\delta_{ss'}\left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{17}{4} + \frac{1}{2}\rho - \frac{5}{6}\pi^2 + \frac{1}{2u_0} + \frac{1}{2}\frac{(1-u_0)(1-3u_0)}{u_0^2}\ln(1-u_0) + 2\mathrm{Li}_2(u_0)\right] + O(\varepsilon)$$

Conclusions and Outlook

- Extension of the dipole formalism to heavy fermions
- Application to top quark physics
- Useful also for SUSY QCD (gluinos)
- Techniques developped here will be useful for NNLO